II.31. (a) Consider the rectangular contour C_R consisting of the intervals $I_R := [-R, R]$, $S_R^+ := [R, R + 2\pi \mathrm{i}/b]$, $J_R := [R + 2\pi \mathrm{i}/b, -R + 2\pi \mathrm{i}/b]$, $S_R^- := [-R + 2\pi \mathrm{i}/b, -R]$. First note that

$$\int_{J_R} \frac{e^{az}\mathrm{d}z}{1+e^{bz}} = -\int_{I_R} \frac{e^{a(x+2\pi \mathrm{i}/b)}\mathrm{d}x}{1+e^{bx}} = -e^{2\pi \mathrm{i}a/b} \int_{I_R} \frac{e^{ax}\mathrm{d}x}{1+e^{bx}}.$$

Next,

$$\left|\int_{S_R^+} \frac{e^{az}\mathrm{d}z}{1+e^{bz}}\right| \le \frac{2\pi}{b} \cdot \frac{e^{\operatorname{Re} aR}}{e^{bR}-1} \to 0 \text{ as } R \to \infty,$$
$$\left|\int_{S_R^-} \frac{e^{az}\mathrm{d}z}{1+e^{bz}}\right| \le \frac{2\pi}{b} \cdot \frac{e^{-\operatorname{Re} aR}}{1-e^{-bR}} \to 0 \text{ as } R \to \infty.$$

So,

$$\int_{-\infty}^{\infty} \frac{e^{ax}\mathrm{d}x}{1+e^{bx}} = \lim_{R\to\infty} \frac{1}{1-e^{2\pi \mathrm{i}a/b}} \left(\int_{I_R} + \int_{J_R}\right) \frac{e^{az}\mathrm{d}z}{1+e^{bz}} = \frac{1}{1-e^{2\pi \mathrm{i}a/b}} \lim_{R\to\infty} \int_{C_R} \frac{e^{az}\mathrm{d}z}{1+e^{bz}}.$$

By the Residue Theorem, the last integral is equal to $2\pi \mathrm{i}$ times the sum of all residues inside the contour. Since the only singularity of the integrand inside the contour is a simple pole at the point $\pi \mathrm{i}/b$, we use the formula $\operatorname{Res}(f/g)(z_0) = \lim_{z\to z_0} f(z)/g'(z)$ to get

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{ax}\mathrm{d}x}{1+e^{bx}} &= \frac{2\pi \mathrm{i}}{1-e^{2\pi \mathrm{i}a/b}} \lim_{z\to\pi \mathrm{i}/b} \frac{e^{az}}{be^{bz}} = \frac{2\pi \mathrm{i}}{1-e^{2\pi \mathrm{i}a/b}} \frac{-e^{\pi \mathrm{i}a/b}}{b} \\ &= \frac{2\pi \mathrm{i}}{b(e^{\pi \mathrm{i}a/b} - e^{-\pi \mathrm{i}a/b})} = \frac{2\pi \mathrm{i}}{b \cdot 2\mathrm{i}\sin(a\pi/b)} = \frac{\pi}{b\sin(a\pi/b)}.\end{aligned}$$

(b) Consider the rectangular contour C_R consisting of the intervals $I_R := [-R, R]$, $S_R^+ := [R, R + \pi \mathrm{i}/a]$, $J_R := [R + \pi \mathrm{i}/a, -R + \pi \mathrm{i}/a]$, $S_R^- := [-R + \pi \mathrm{i}/a, -R]$. First note that

$$\int_{J_R} \frac{\sinh az \,\mathrm{d}z}{\sinh 4az} = -\int_{I_R} \frac{\sinh(a(x+\pi \mathrm{i}/a))\,\mathrm{d}x}{\sinh(4a(x+\pi \mathrm{i}/a))} = \int_{I_R} \frac{\sinh ax \,\mathrm{d}x}{\sinh 4ax}.$$

Also,

$$\left|\int_{S_R^\pm} \frac{\sinh az \,\mathrm{d}z}{\sinh 4az}\right| \le \frac{\pi}{a} \cdot \frac{|e^{aR} - e^{-aR}|}{|e^{4aR} - e^{-4aR}|} \to 0 \text{ as } R \to \infty.$$

So,

$$\int_{-\infty}^{\infty} \frac{\sinh ax \,\mathrm{d}x}{\sinh 4ax} = \lim_{R\to\infty} \frac{1}{2} \left(\int_{I_R} + \int_{J_R}\right) \frac{\sinh az \,\mathrm{d}z}{\sinh 4az} = \frac{1}{2} \lim_{R\to\infty} \int_{C_R} \frac{\sinh az \,\mathrm{d}z}{\sinh 4az}.$$

The singularities of the integrand inside the contour are simple poles at $\pi \mathrm{i}/4a$ and $3\pi \mathrm{i}/4a$ and a double pole at $\pi \mathrm{i}/2a$. (Incidentally, since the limits $\lim_{z\to 0} \sinh az/\sinh 4az$, $\lim_{z\to\pi \mathrm{i}} \sinh az/\sinh 4az$ exist, the singularities on the contour itself are removable.) Note also that the residue at

$\pi \mathrm{i}/2a$ is zero, since the Taylor expansion of $\sinh 4az$ around the point $\pi \mathrm{i}/2a$ begins with a term of order 2. Thus,

$$\int_{-\infty}^{\infty} \frac{\sinh ax \, \mathrm{d}x}{\sinh bx} = \frac{2\pi \mathrm{i}}{2} \sum_{j=1,3} \lim_{z \to j\pi \mathrm{i}/4a} \frac{\sinh az}{4a \cosh 4az} = \frac{\pi \mathrm{i}}{4a} \left(\frac{\mathrm{i} \sin(\pi/4)}{\cos \pi} + \frac{\mathrm{i} \sin(3\pi/4)}{\cos 3\pi} \right) = \frac{\sqrt{2}\pi}{4a}.$$

III.1. Let v be the vehicle's speed, m its mass, and α the air resistance coefficient. Then, by Newton's second law,

$$m\dot{v} = mF - \alpha v.$$

The characteristic equation for the corresponding homogeneous ODE is

$$m\lambda + \alpha = 0,$$

hence the general solution of the homogeneous equation is $v_h(t) = Ce^{-\alpha t/m}$. A particular solution to the inhomogeneous equation is $v_i(t) = mF/\alpha$. Hence the general solution is

$$v_g(t) = \frac{mF}{\alpha} + CE^{-\alpha t/m}.$$

The initial condition $v(0) = 0$ now implies $C = -mF/\alpha$. Hence the solution to the problem is

$$v(t) = \frac{mF}{\alpha} \left(1 - e^{-\alpha t/m} \right).$$

III.2 The characteristic equation for this ODE is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0, \quad \text{i.e.,} \quad (\lambda - 1)^3 = 0.$$

So, $\lambda = 1$ is a root of multiplicity 3, hance the general solution is obtained by multiplying an arbitrary polynomial of degree 2 by the exponential e^x:

$$y(x) = (A + Bx + Cx^2)e^x, \qquad \text{where } A, B, C \text{ are constants.}$$

III.3 According to the description of the circuit,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0,$$
$$q(0) = C\mathcal{E}, \quad \frac{dq}{dt}(0) = 0.$$

The characteristic equation of this ODE is

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0.$$

Its roots are

$$\alpha := -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}, \quad \beta := -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}.$$

If $\alpha \neq \beta$, the general solution to the ODE has the form $Ae^{\alpha t} + Be^{\beta t}$, so to match the initial conditions, we must have

$$A + B = C\mathcal{E}, \qquad A\alpha + B\beta = 0,$$

from which we get $A = -C\mathcal{E}\beta/(\alpha - \beta)$, $B = C\mathcal{E}\alpha/(\alpha - \beta)$. If $\alpha = \beta$, then the general solution is $(A + Bt)e^{\alpha t}$, and the initial condition yields $A = C\mathcal{E}$, $A\alpha + B = 0$, so $B = -C\mathcal{E}\alpha$.

III.12 Multiply both sides of the equality

$$\frac{y''}{y'} = -\frac{y'}{y}$$

by dx and integrate, taking into account the facts $\mathrm{d}(y') = y''\mathrm{d}x$, $\mathrm{d}y = y'\mathrm{d}x$. We get

$$\int \frac{\mathrm{d}y'}{y'} = -\int \frac{\mathrm{d}y}{y} \Longleftrightarrow \ln y' = -\ln y + C.$$

Therefore $2y' = C_1/y$, i.e., $2yy' = C_1$, hence noticing that $2yy' = (y^2)'$ and integrating again, we get the general solution (in implicit form)

$$y^2 = C_1 x + C_2, \text{ where } C_1, C_2 \text{ are constants.}$$

III.13 First look for a solution in the form

$$y(x) := \sum_{n=-\infty}^{\infty} a_n x^n. \tag{1}$$

Then

$$y'(x) = \sum_{n=-\infty}^{\infty} n a_n x^{n-1}, \qquad y''(x) = \sum_{n=-\infty}^{\infty} n(n-1) a_n x^{n-2},$$

so

$$y'' - y' + \frac{y}{x} = \sum_{n=-\infty}^{\infty} \left((n+1)n a_{n+1} - n a_n + a_n\right) x^{n-1} = 0.$$

Each coefficient of this series must equal zero. Taking $n = 0$, we see that $a_0 = 0$; taking $n = -1$, we get $2a_{-1} = 0$, hence $a_{-1} = 0$, and then for any $n \leq -2$ we get $a_n = n(n+1)a_{n+1}/(n-1)$, so $a_n = 0$. Likewise, taking $n = 1$, we get $a_2 = 0$, and then $a_n = 0$ for all $n \geq 2$. Therefore, the only solution in the form (1) is $y = a_1 x$ where a_1 is an arbitrary constant.

To find the second solution linearly independent of the first, look for a solution in the form

$$y(x) := \sum_{n=-\infty}^{\infty} a_n x^n + cx \ln x. \tag{2}$$

Then

$$y'(x) = \sum_{n=-\infty}^{\infty} n a_n x^{n-1} + c \ln x + c, \qquad y''(x) = \sum_{n=-\infty}^{\infty} n(n-1) a_n x^{n-2} + \frac{c}{x},$$

so

$$y'' - y' + \frac{y}{x} = \sum_{n=-\infty}^{\infty} \left((n+1)na_{n+1} - na_n + a_n\right) x^{n-1} + \frac{c}{x} - c = 0.$$

Taking $n = -1$, we still get $a_{-1} = 0$, which implies that $a_n = 0$ for all $n \leq -1$. On the other hand, taking $n = 0$, we now get $a_0 + c = 0$; taking $n = 1$, we get $2a_2 - c = 0$; taking $n \geq 2$, we get $a_{n+1} = (n-1)a_n/n(n+1)$. Hence, the other solution has the form

$$y(x) = c\left(x \ln x - 1 + \sum_{n=2}^{\infty} \frac{(n-2)!!}{n((n-1)!)^2} x^n\right).$$

Here $m!!$ denotes the product $\prod_{j=0}^{\lfloor m/2 \rfloor}(m-2j)$.