IV.2.

(a) $f(x) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n}$. (b) $-\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{\sin(\pi n x)}{n}$.

(c) The series for $[0,1]$ converges faster to the function f on $[0,1]$ than the series for $[-1,1]$. On the other hand, the second series approximates the function on a larger interval.

IV.10.

$$\frac{3}{4} + \sum_{n=1}^{\infty}\left(-\frac{1}{\pi n}\sin(\pi n x) + \frac{(-1)^n - 1}{\pi^2 n^2}\cos(\pi n x)\right).$$

IV.13. The Fourier series of f is

$$f(t) = -\sum_{p=0}^{\infty} \frac{2aT}{\pi^2(2p+1)^2}\cos\left(\frac{(4p+2)\pi t}{T}\right).$$

With

$$\alpha := \frac{-a + \sqrt{a^2 - 4k^2}}{2}, \qquad \beta := \frac{-a - \sqrt{a^2 - 4k^2}}{2},$$

the solution to the given ODE can be written as

$$Ae^{\alpha t} + Be^{\beta t} + \sum_{n \in \mathbb{N},\, n \text{ odd}} \left(a_n \sin\left(\frac{2\pi n t}{T}\right) + b_n \cos\left(\frac{2\pi n t}{T}\right)\right),$$

$$a_n = \frac{2aT}{\pi^2 n^2 \sqrt{a^2 - 4k^2}}\left(\frac{\alpha}{\alpha^2 + 4\pi^2 n^2/T^2} - \frac{\beta}{\beta^2 + 4\pi^2 n^2/T^2}\right),$$

$$b_n = \frac{4a}{\pi n \sqrt{a^2 - 4k^2}}\left(\frac{1}{\alpha^2 + 4\pi^2 n^2/T^2} - \frac{1}{\beta^2 + 4\pi^2 n^2/T^2}\right).$$

IV.16.

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n x}{L}\right)\cos\left(\frac{\pi n v t}{L}\right),$$

$$a_n = \frac{aL^2}{\pi^3 n^3}\left(3\cos(\pi n/3) + 2\pi n \sin(\pi n/3) + \cos(\pi n) - 4\right).$$

IV.19. The Fourier series for x^2 on the interval $[-\pi, \pi]$ is

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx.$$

Taking $x := 0$, we arrive at

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}.$$

Taking $x := \pi$, we get

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Finally, by Parseval,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

IV.20. The odd extension of the constant 1 on the interval $[0, \pi]$ to the interval $[-\pi, \pi]$ has the Fourier series

$$\sum_{n=1,\, n \text{ odd}} \frac{4 \sin nx}{\pi n}.$$

Taking $x := \pi/2$, we get

$$\frac{\pi}{4} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)}.$$

Parseval gives

$$\frac{\pi^2}{8} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}.$$