Sample Midterm Questions - Math 128B

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1a. (*points*) In the following table, for each 'example differential equation' described in the first 2 give a classification (e.g., "inhomogeneous linear 1st order IVP") in the third column and suggest a numerical method for solving it.
Note $\mathbf{y}(t) = (y_1(t), y_2(t))$ is denoted simply (y_1, y_2).

ODE	Constraints	Classification, Numerical method
$x''(t) = 3t^2x'(t) - e^{-t^2}x(t) + 7$	$x(0) = 0, x(1) = 1$	?
$\mathbf{y}'(t) = (y_2 + y_1, y_2^2)$	$\mathbf{y}(0) = (0, 1), \mathbf{y}(1) = (2, 3)$	?
$\mathbf{y}'(t) = (\cos(t)y_2 - \sin(t)y_1, 1)$	$\mathbf{y}(0) = (0, 1), \mathbf{y}'(0) = (2, 3)$	?
$\mathbf{y}'(t) = (ty_2 + y_1, y_1^2)$	$\mathbf{y}(0) = (0, 1), \mathbf{y}'(0) = (0, 1)$	?

2. Consider the Heat Equation defined on the interval [0,1]:

$$u_t(x,t) = u_{xx}(x,t) \qquad x \in [0,1] \qquad t \geq 0 \qquad u(0,t) = u(1,t) = 0$$

2a. (*points*) If $u(x,t)$ is not specified at $t = 0$, show that the above PDE is under-determined (i.e. uniqueness fails).

Hint: Look for polynomial functions in x and t and show there are an infinite number of such solutions satisyfing both the PDE and the boundary conditions.

2b. (*points*) If $u(x,t)$ is specified at $t = 0$, AND ALSO $u_t(x,t)$ is specified at $t = 0$ (as is done in the Wave Equation) then *prove* that the above PDE can be over-determined (i.e. existence fails).

Hint: Assume $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ are specified. What relationship between f and g must be enforced in order for this initial data to be consistent with the PDE?

3. Consider the 2nd order BVP for $y(x)$:

$$y''(x) = f(x, y, y'), \quad y(x = a) = \alpha, \quad y(x = b) = \beta$$

By considering instead a corresponding IVP:

$$y''(x) = f(x, y, y'), \quad y(x = a) = \alpha, \quad y'(x = a) = t$$

Burden & Faires develop the Nonlinear Shooting Method described on pages 654-656, where the following equations are written (Note that x is the independent variable, and t is initial slope - not to be confused with '$t =$ independent variable' as used in Mathews & Fink):

$$y(b, t) - \beta = 0 \tag{11.8}$$

$$t_k = t_{k-1} - \frac{y(b, t_{k-1}) - \beta}{\frac{dy}{dt}(b, t_{k-1})} \tag{11.9}$$

$$z(x, t) \stackrel{\text{def}}{=} (\partial y/\partial t)(x, t)$$

$$z''(x, t) = \frac{\partial f}{\partial y}(x, y, y')z(x, t) + \frac{\partial f}{\partial y'}(x, y, y')z'(x, t)$$

$$a \le x \le b, z(a, t) = 0, z'(a, t) = 1 \tag{11.12}$$

$$t_k = t_{k-1} - \frac{y(b, t_{k-1}) - \beta}{z(b, t_{k-1})} \tag{11.13}$$

In the particular case of the BVP:

$$y''(x) = -25y, \quad y(x = 0) = 1, \quad y(x = \pi/2) = 100$$

1. State the corresponding IVP (with unknown initial velocity $y'(0) = t$).
2. Give an exact solution $y(x, t)$ to the above IVP (explicitly showing its dependence on the unknown initial velocity).
3. What equation (involving only t) does (11.8) become?
4. What equation (involving only t) does (11.12) become?
5. Give an exact solution $z(x, t)$ to (11.12).
6. What equation (involving only t_k and t_{k-1}) does (11.13) become?
7. Using the above equation for t_k, determine $t = \lim_{k\to\infty} t_k$.
8. Since the above BVP is linear, the value of t determined above must be consistent with the initial velocity which results from the Linear Shooting Algorithm - prove this.

4. On the bottom of page 668 it's explained that "Newton's method for nonlinear systems requires that at each iteration the NxN linear system ...

$$J(w_1, \ldots, w_N)(v_1, \ldots, v_N)^t = -(\ldots)^t$$

be solved for $v_1, v_2, \ldots, v_N$, since

$$w_i^{(k)} = w_i^{(k-1)} + v_i, \text{ for each } i = 1, 2, \ldots, N."$$

4a. (*points*) Justify the above statement.

4b. (*points*) In the special case when the original ODE is linear, prove that the above Newton iteration converges in one step.

Hint: What does $J(w_1, \ldots, w_N)$ simplify to in the linear case?