Homework # 12, due Mon, Nov 28th.

1. Each of the numbers x_1, x_2, …, x_n is either 1 or -1. If the sum

$$S := x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + \cdots + x_nx_1x_2x_3 = 0,$$

prove that n must be a multiple of 4.

2. Prove that

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$$

for all positive integers $n \geq 2$.

3. Find all real-valued continuously differentiable functions on the real line such that, for all x,

$$(f(x))^2 = \int_0^x \left((f(t))^2 + (f'(t))^2\right) \mathrm{dt} + 2004.$$

4. Let a_0, …, a_n be arbitrary numbers and $A_k := \sum_{j \leq k} a_j$. Prove that

$$\begin{vmatrix} A_0 & A_0 & A_0 & A_0 & \cdots & A_0 \\ A_0 & A_1 & A_1 & A_1 & \cdots & A_1 \\ A_0 & A_1 & A_2 & A_2 & \cdots & A_2 \\ A_0 & A_1 & A_2 & A_3 & \cdots & A_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_0 & A_1 & A_2 & A_3 & \cdots & A_n \end{vmatrix} = a_0a_1a_2\cdots a_n.$$

5. Evaluate the integral

$$\int_0^\infty e^{-x}\ln^2 x \,\mathrm{dx}.$$

6. Let k be a positive integer. For which values of the real number c does the differential equation

$$\frac{d^2x}{dt^2} - 2c\frac{dx}{dt} + x = 0$$

have a nontrivial solution satisfying $x(0) = x(2\pi k) = 0$?

7. Prove that

$$\int_0^\infty \frac{\cos(x^2) - \cos x}{x}\mathrm{dx} = \frac{1}{2}\gamma,$$

where γ is the Euler-Mascheroni constant.

8. For a permutation $a := (a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$, let

$$S(a) := (a_1 - a_2)^2 + (a_2 - a_3)^2 + \cdots + (a_{n-1} - a_n)^2.$$

what is the average value of S taken over all permutations?