

## Inclusion-exclusion principle.

**Inclusion-exclusion formula.** Suppose  $P_1, \dots, P_n$  are subsets of  $S$ . Let

$$M_k := \sum_{|I|=k} M(\supseteq I) \quad \text{where} \quad M(\supseteq I) := |\cap_{i \in I} P_i|.$$

Then the number of elements of  $S$  that belong to precisely  $r$  of the sets  $P_j$  is given by

$$E_r = \sum_{k=r}^n (-1)^{k-r} \binom{k}{r} M_k. \tag{1}$$

**Proof.** Note first that

$$M_k = \sum_{r=k}^n \binom{n}{k} E_r$$

since an element belonging to exactly  $r$  of the  $P_j$ 's contributes  $\binom{r}{k}$  times to the sum defining  $M_k$ . Consider polynomials

$$M(x) := \sum_{k=0}^n M_k x^k, \quad E(x) := \sum_{r=0}^n E_r x^r.$$

Then

$$M(x) = \sum_{k=0}^n \sum_{r=k}^n \binom{r}{k} E_r x^k = \sum_{r=0}^n E_r \sum_{k=0}^r \binom{r}{k} x^k = \sum_{r=0}^n E_r (x+1)^r = E(x+1).$$

It follows that

$$E(x) = M(x-1) = \sum_{k=0}^n M_k (x-1)^k = \sum_{k=0}^n M_k \sum_{r=0}^k \binom{k}{r} x^r (-1)^{k-r} = \sum_{r=0}^n x^r \sum_{k=r}^n (-1)^{k-r} \binom{k}{r} M_k,$$

from which we get (1).

**Derangements.** Let  $D_n$  denote the number of permutations of  $[n]$  with no fixed points. Then

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}.$$

**Proof.** There are  $\binom{n}{k}$  choices for  $I \subseteq [n]$  with  $|I| = k$ , and for each one there are  $(n-k)!$  permutations of  $[n]$  such that  $\pi(i) = i$  for all  $i \in I$ . Thus  $M_k = \binom{n}{k} (n-k)! = n!/k!$ . So,

$$D_n = \sum_{k=0}^n (-1)^k M_k = n! \sum_{k=0}^n \frac{(-1)^k}{k!}.$$

More generally, the number of permutations of  $[n]$  having exactly  $r$  fixed points is given by

$$E_r = \sum_{k=r}^n (-1)^{k-r} \binom{k}{r} \frac{n!}{k!} = \frac{n!}{r!} \sum_{j=0}^{n-r} \frac{(-1)^j}{j!} = \binom{n}{r} D_{n-r}.$$

**Examples.**

1. Show that the number of permutations  $\pi$  of  $[n]$  such that  $\pi(j+1) \neq \pi(j) + 1$  for  $j = 1, \dots, n-1$  is  $D_n + D_{n-1}$ .
2. Prove that

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n.$$

3. How many  $m \times n$  0-1 matrices have no row or column consisting entirely of zeros?
4. Find the generating function

$$G_n := \sum_{\sigma \in S_n} x^{w(\sigma)}$$

where  $S_n$  denotes the set of all permutations of  $[n]$  and  $w(\sigma)$  is the number of fixed points of  $\sigma$ .

5. Using the inclusion-exclusion formula, show that

$$\sum_{U \subseteq S} (-1)^{|U|} \binom{m - \sigma(U)}{|S|} = \pi(S) \quad \text{for all } m \geq \sigma(S),$$

where  $|U|$ ,  $\sigma(U)$ , and  $\pi(U)$  denote the number of elements, the sum and the product, respectively, of a finite set  $U$ . (For the empty set, these numbers are 0, 0, 1.)