Answers to homework assignment IX

VI.8 (ii) $\Longrightarrow$ (i): $b^t y = p^t(Ay) \ge 0$ if both p and Ay are nonnegative.

not (ii) $\Longrightarrow$ not (i): If b fails to be in the closed convex set $\{p^t A : p \ge 0\}$ (why is this set closed?), then, by (13)Corollary, can find y with $\inf_{p\ge 0} p^t Ay > b^t y$. In particular, $Ay \ge 0$ (since, otherwise, the inf would be $-\infty$), and, with that, the inf is 0, hence $b^t y < 0$.

VI.9 Let $x \in \cap_i K_i$. If, for all i, $\inf \lambda_i(K_i) \ge 0$ for some $\lambda_i \in X^*$, then $\lambda_i(x) \ge 0$, all i, hence, with $\lambda := \sum_i \lambda_i$, also $\lambda x \ge \lambda_i x$, all i. Assume now, by way of contradiction, that some $\lambda_i \ne 0$ while $\lambda = \sum_i \lambda_i = 0$. Then $\lambda_i \ne 0$ for some $i > 0$, and, since K_i is open, there is $\varepsilon > 0$ so that $B_\varepsilon(x) \subset K_i$, hence, $\lambda x \ge \lambda_i x > \lambda_i x - \|\lambda_i\|\varepsilon = \inf \lambda_i B_\varepsilon(x) \ge \inf \lambda_i(K_i) \ge 0$, leading to the contradiction $\lambda \ne 0$.

Assume that $\cap_i K_i = \{\}$. Then $K := \{(x, \ldots, x) \in X^n : x \in K_0\}$ is a convex set in the nls X^n that has no intersection with the nonempty open convex set $L := K_1 \times \cdots \times K_n \subset X^n$. By (12)Corollary, there is $0 \ne \lambda = (\lambda_1, \ldots, \lambda_n) \in (X^n)^* = (X^*)^n$ for which $0 \le \sup(\lambda_1 + \cdots + \lambda_n)(K_0) = \sup \lambda(K) \le \inf \lambda(L) = \sum_i \inf \lambda_i(K_i) \le \sum_i 0 = 0$, the first and last inequality since $0 \in K_i^-$ for all i. Therefore, with $\lambda_0 := -(\lambda_1 + \cdots + \lambda_n)$, we have $\sum_{i=0}^n \lambda_i = 0$ while $\inf_i \lambda_i(K_i) = 0$, all i.

VI.10 By H.P.(V.6), any weakly closed set is (norm-)closed. Conversely, if K is convex and (norm-)closed and $x \notin K$, then, by (13)Corollary, $r := \lambda x - \sup \lambda(K) > 0$ for some $\lambda \in X^*$, hence the weak nbhd $B_{r,\lambda}(x)$ fails to intersect K, i.e., x is not in the weak closure of K, either.

VI.13 Taking for granted that any n-dimensional lss Y of $C(T)$ over $\mathbb{C}$ has dimension $2n$ as a lss over $\mathbb{R}$, and observing that now every lfl of the form $e^{i\alpha}\delta_t$ (for $t \in T$ and $\alpha \in \mathbb{R}$) is an extreme point for the closed unit ball of the dual, adaptation of the proof of (18) gives in the case of a blfl λ on the n-dimensional lss Y of $C(T)$ of complex-valued functions the existence of $U \subset T$ with $\#U \le 2n$ and $a \in \mathbb{R}^U$ and $b \in \mathbb{C}^U$ with $|b(u)| = 1$ for all u so that $\lambda = \sum_{u\in U} a(u)b(u)\delta_u$ on Y while $\|\lambda\| = \|a\|_1$. Since an arbitrary $z \in \mathbb{C}$ can be written as $z = ae^{i\alpha}$, we can simplify this to: there exists $U \subset T$ with $\#U \le 2n$ and $a \in \mathbb{C}^U$ so that $\lambda = \sum_{u\in U} a(u)b(u)\delta_u$ on Y while $\|\lambda\| = \|a\|_1$.

VI.14 The space $Y + \mathrm{ran}[1]$ has dimension $\le n + 1$, hence, by (20)Proposition, there is such a quadrature rule using at most $n + 1$ points.

VI.15 The Alternation Theorem uses crucially the Haar property. Here is an example of what may happen without it: $X = C([-1\,..\,1])$, $Y = \mathrm{ran}[()_+]$, $x = ()^0$. Then, for any $\alpha \in [0\,..\,2]$, $\alpha()_+$ is a ba to x from Y. In particular, $y = 0$ is a ba, yet the error, $x - y = x$, is positive throughout and, in particular, fails to even change sign, let alone alternate.