# Solutions to homework #1.

1. First note that $a \equiv b \pmod m$ implies $f(a) \equiv f(b) \pmod m$ for integers $a$, $b$, $m$, and a polynomial with integer coefficients $f$. Indeed, if $f(x) = \sum_{j=0}^n a_j x^j$, $a_j \in \mathbb{Z}$, $j = 0, \ldots, n$, then $f(a) - f(b) = \sum_{j=0}^n a_j(a^j - b^j) = (a-b)\sum_{j=0}^n a_j(a^{j-1} + a^{j-2}b + \cdots + ab^{j-2} + b^{j-1}) \equiv 0 \pmod m$.

The numbers 1 through $k$ form a complete set of representatives modulo $k$, so if $f(j) \not\equiv 0 \pmod k$ for all $j = 1, \ldots, k$, then, by the earlier observation, $f$ has no integer root modulo $k$, in particular, it has no integer root.

2. There are no such functions. Indeed, multiply the first equality by $\alpha^2$, the second by $(-2\alpha)$, and add the two modified equalities to the third. We get:

$$\int_0^1 f(x)(x-\alpha)^2 \mathrm{dx} = 0.$$

Since the integrand is nonnegative and continuous, the integral is zero if and only if the integrand is zero on the whole interval $[0,1]$, which contradicts the positivity of $f$.

**Remark.** The conclusion holds even if continuity of $f$ is relaxed to Lebesgue integrability, since then $f$ still has to be zero almost everywhere in $[0,1]$, and this contradicts the first condition $\int_0^1 f(x)\mathrm{dx} = 1$.

3. **Answer:** 18. The inequality $x^4 + 36 \le 13x^2$ is equivalent to $(x^2-4)(x^2-9) \le 0$, which is equivalent to $2 \le |x| \le 3$. Since $f'(x) = 3(x^2-1) > 0$ whenever $2 \le |x| \le 3$, the function $f$ increases on both intervals $[-3,-2]$ and $[2,3]$. In addition, $f$ is an odd function. Hence its maximum is attained at the point 3 and is equal to $f(3) = 18$.

4. Since

$$\sum_{j=0}^n \binom{n}{j} x^j \sum_{j=0}^n \binom{n}{j} x^{n-j} = ((1+x)^n)^2 = (1+x)^{2n} = \sum_{j=0}^n \binom{2n}{j} x^j,$$

the coefficient of $x^n$ in the right-hand side, $\binom{2n}{n}$, is equal to the coefficient of $x^n$ in the left-hand side, which equals $\sum_{j=0}^n \binom{n}{j}^2$.

5. **Answer:** $f^{(k)} = 0$ if $k$ is odd, $f^{(k)}(0) = (-1)^{k/2}k!$ if $k$ is even. Let $g(x) := 1/(1+x^2)$. Then $f(1/n) = g(1/n)$ for all $n \in \mathbb{N}$. Expanding $g$ as a geometric series, we see that

$$1 - x^2 + x^4 - \cdots + (-x^2)^n + \cdots$$

is the Taylor expansion of $g$ near the origin, hence $g^{(k)} = 0$ if $k$ is odd, $g^{(k)}(0) = (-1)^{k/2}k!$ if $k$ is even. Therefore, it remains to show that all derivatives of the function $f - g$ are zero at the origin.

**Lemma.** Suppose $h$ is an infinitely differentiable real-valued function defined on $\mathbb{R}$ such that $h(1/n) = 0$ for all $n \in \mathbb{N}$. Then $h^{(k)}(0) = 0$ for all $k \in \mathbb{Z}_+$.

Proof. Since $h$ is infinitely differentiable in a neighborhood of 0, the $k$th derivative of $h$ at 0 is the limit of its normalized $(k+1)$th divided difference at distinct nodes $x_1$, …, $x_{k+1}$ as they tend to 0:

$$h^{(k)}(0) = k! \lim_{x_1,\ldots,x_{k+1}\to 0} \Delta(x_1,\ldots,x_{k+1})h.$$

Now, choosing $x_j := x_j(n) = 1/(n+j)$ and letting $n \in \mathbb{N}$ tend to infinity, we see that $h^k(0) = 0$.

6. **Answer:** $x^{n-1}(x + a_1^2 + \cdots + a_n^2)$. Let $a := [a_1\ a_2\ \ldots\ a_n]^T$. The determinant we need to evaluate is the characteristic polynomial of the matrix $aa^T$ at $-x$. If $a = 0$, the polynomial is simply $x^n$.

Otherwise any vector $b$ orthogonal to $a$ (so that $a^Tb = 0$) is annihilated by the matrix $aa^T$. The space of such vectors is $(n-1)$-dimensional. The vector $a$ itself is an eigenvector corresponding to eigenvalue $a^Ta$: $aa^Ta = (a^Ta)a$. So, the space $\mathbb{R}^n$ consisting of eigenvectors of $aa^T$, $n-1$ corresponding to eigenvalue 0 and 1 corresponding to eigenvalue $a^Ta$. Therefore, the characteristic polynomial of $aa^T$ is $(-x)^{n-1}(a^Ta - x)$, and the answer is $x^{n-1}(a^Ta + x)$.

7. There are $8 \cdot 9^{n-1}$ $n$-digit numbers that do not contain digit 9, as there are 8 choices for the leading digit and 9 for all other digits. Each such number is at least $10^{n-1}$ so the sum of their reciprocals is at most $8 \cdot (9/10)^{n-1}$. Therefore the series under consideration is majorized by the converging geometric series

$$\sum_{n=1}^{\infty} 8 \cdot \left(\frac{9}{10}\right)^{n-1},$$

hence it converges as well.

8. The statement is false. Let $g$ be any continuous function on $\mathbb{R}$ supported on $[0,1]$ with, say, $\max_{x\in[0,1]} g(x) = 1$. Define

$$f(x) := \begin{cases} g(\{x\}\lfloor x\rfloor) & x > 0 \\ 0 & x \le 0. \end{cases}$$

Then $f$ is supported on the union of intervals $\cup_{n\in\mathbb{N}}[n, n+\frac{1}{n}]$. It is continuous in each open interval $(n, n+1)$, $n \in \mathbb{N}$, as the composition of continuous functions and at each point $n \in \mathbb{N}$ since $g$ is continuous at 0 and 1 and $g(0) = g(1) = 0$. Thus, $f$ is continuous everywhere in $\mathbb{R}$. For all $\varepsilon > 0$, $f(\varepsilon + n) = 0$ for all $n \ge \frac{1}{\varepsilon}$), hence $\lim_{n\to\infty} f(\varepsilon + n) = 0$. But $\max_{x\in[n,n+1]} f(x) = 1$ for all $n \in \mathbb{N}$, so $\lim_{x\to\infty} f(x) \neq 0$.

9. First of all, since the integrand is continuous on $[0,1]$, the integral $\int_0^1 \sin x \sin x^2 \mathrm{dx}$ is finite, so it is enough to prove that the integral

$$\int_1^B \sin x \sin x^2 \mathrm{dx}$$

converges. Multiplying and dividing the integrand by $2x$, we get

$$\int_1^B \frac{\sin x}{2x}(-\mathrm{d}(\cos x^2)) = -\left.\frac{\sin x}{2x}\cos x^2\right|_1^B + \int_1^B \left(\frac{\cos x}{2x} - \frac{\sin x}{2x^2}\right)\cos x^2 \mathrm{dx}.$$

The function $\frac{\sin x \cos x^2}{2x}$ tends to 0 as $x$ tends to infinity, abd the integral of $\frac{\sin x}{2x^2}\cos x^2$ converges absolutely as $B \to \infty$ by comparison to the integral of $1/x^2$. It remains to consider

$$\int_1^B \frac{\cos x}{2x}\cos x^2 \mathrm{dx} = \int_1^B \frac{\cos x}{4x^2}\mathrm{d}(\sin x^2) = \frac{\cos x}{4x^2}\sin x^2\Big|_1^B - \int_1^B \frac{-2\cos x - x\sin x}{4x^3}\sin x^2 \mathrm{dx}.$$

Now $\frac{\cos B}{4B^2}\sin B^2 \to 0$ as $B \to \infty$ and the last integral converges absolutely by comparison with the integral of $1/x^2$.

10. **Answer:** 1/8. Pick the first three points first; call them $A$, $B$, $C$. Consider the spherical triangle $T$ defined by those points. The center of the sphere lies in the convex hull of $A$, $B$, $C$ and another point $P$ if and only if $P$ is antipodal to some point in $T$. So the desired probability is the expected fraction of the sphere's surface area which is covered by $T$.

Given a point $P$, denote its antipode by $P'$. Consider 8 spherical triangles: $ABC$, $ABC'$, $AB'C$, $AB'C'$, $A'BC$, $ABC''$, $A'B'C$, $A'B'C'$. They cover the sphere with overlap of measure zero, since they are the 8 regions formed by the three great circles obtained by extending the sides of sperical triangle $ABC$. Since the map $(A,B,C) \mapsto (A',B,C)$ is an automorphism of our probability space, all 8 triangles are equally distributed functions of random variables $A$, $B$, $C$. In particular, the expected fraction of the covered surface area of the sphere is the same for each triangle. Since the total area is 1, each expected fraction is 1/8.

11. **Answer:** $\lceil p/4 \rceil$. Let $S$ be the set of solutions to $x^2 = y^2 + 1$ over $\mathbb{Z}_p$. The linear change of coordinates $(x,y) \mapsto (x+y, x-y)$ is invertible since its determinant is $-2 \neq 0$ in $\mathbb{Z}_p$. Hence $|S|$, the cardinality of $S$, equals the number of solutions to $uv = 1$ over $\mathbb{Z}_p$. There is exactly one possible $v$ for each nonzero $u$ and no $v$ for $u = 0$, so $|S| = p - 1$.

The problem asks for the size of the image of the map $\phi : S \to \mathbb{Z}_p$ taking $(x,y)$ to $x^2$. If $z = x^2$ for some $(x,y) \in S$, then the full preimage $\phi^{-1}(z)$ is the set $\{(\pm x, \pm y)\}$, which has size 4, except in the cases $z = 1$ ($x = \pm 1$, $y = 0$, making $\phi^{-1}(z)$ of size 2) and $z = 0$ (in which case $x = 0$ and $y^2 = -1$, again making $\phi^{-1}(z)$ of size 2); the latter case is possible only if $-1$ is a square in $\mathbb{Z}_p$. Hence $|S| = 4|\phi(S)| - 2 - 2c$ where $c$ is 1 or 0 according to whether or not $-1$ is a square. Thus

$$|\phi(S)| = \frac{(p-1)+2+2c}{4} = \frac{p+1+2c}{4} = \left\lceil \frac{p}{4} \right\rceil.$$

(The penultimate number has to be an integer, so $c = 1$ if $p \equiv 1 \pmod 4$ and 0 if $p \equiv -1 \pmod 4$.)

12. Let $M := \sum_{j=1}^r M_j$. Since multiplication by an element of the group only permutes the sequence $(M_j)$, the product $M_j M$ equals $M$ for all $j = 1, \ldots, r$. Therefore $M^2 = \sum_{j=1}^r M_j M = rM$. Since the polynomial $f(x) := x^2 - rx = x(x-r)$ annihilates $M$, it also annihilates any eigenvalue of $M$. Therefore, $M$ can have only 0 or $r$ as an eigenvalue. However, if at least one of the eigenvalues were equal to $r$, then $\mathrm{tr}M \geq r$, which would contradict the assumption that $\mathrm{tr}M = 0$. Hence $M$ has only zero eigenvalues and the matrix $M - rI$ is invertible. But then $0 = (M^2 - rM)(M - rI)^{-1} = M$.