Solutions to homework #2.

1. **Answer:** $3\sqrt{2}/(3+\sqrt{2})$. Consider the cross-section through the vertex A of the cone and a diagonal BC of the top of the cube. We get two similar triangles ABC and $AB'C'$ where $B'C'$ is a diameter of the base of the cone. If the side-length of the cube is denoted by x, then the diagonal BC has length $\sqrt{2}x$. The height of $AB'C'$ is the height of the cone, and the height of the ABC is $3-x$, since the distance between BC and $B'C'$ is the side-length of the cube. So,

$$\frac{\sqrt{2}x}{2} = \frac{3-x}{3}, \quad \text{hence } x = \frac{3\sqrt{2}}{3+\sqrt{2}}.$$

2. For simplicity of notation, adopt the convention that all indices are considered mod n, so that $n+j$ is identified with j and $-j$ with $n-j$. Then there are n terms of the form $m_j := x_j x_{j+1} x_{j+2} x_{j+3}$, each ± 1, summing up to 0. This is possible only if n is even, and there are as many $+1$'s as there are -1's among the products m_j. The product $\prod_{j=1}^n m_j$ is therefore $(-1)^{n/2} 1^{n/2} = (-1)^{n/2}$. On the other hand, it contains each x_j exactly 4 times, hence is equal to 1. So, $n/2$ must be even as well.

3. Let $\omega := e^{\mathrm{i}2\pi/n}$. Since $\{\omega^j : j = 1, \ldots, n-1\}$ is the set of all nth roots of unity with the exception of 1, we get

$$\frac{z^n - 1}{z-1} = z^{n-1} + z^{n-2} + \cdots + 1 = (z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1}). \tag{1}$$

Since $\omega = \varepsilon^2$ where $\varepsilon := e^{\mathrm{i}\pi/n}$, we get

$$|1-\omega^j| = |\overline{\varepsilon}^j - \varepsilon^j| = 2\sin\frac{j\pi}{n},$$

so evaluating (1) at $z=1$ and taking absolute values of both sides yields the desired result:

$$n = 2^{n-1} \prod_{j=1}^{n} \sin\frac{j\pi}{n}.$$

4. **Answer:** $f(x) = \pm\sqrt{2004}e^x$. Differentiate both sides to obtain

$$2f(x)f'(x) = (f(x))^2 + (f'(x))^2,$$

which is equivalent to $(f(x) - f'(x))^2 = 0$. Since f is real-valued, the latter is possible if and only if $f(x) = f'(x)$, hence $f(x) = Ce^x$ where C is a constant. Plugging this function into the original equation, we get

$$C^2 e^{2x} = C^2 e^{2x} - C^2 + 2004.$$

So, $C = \pm\sqrt{2004}$, and we get two solutions $y = \pm\sqrt{2004}e^x$.

5. The function $x \mapsto \pi(x)$ takes on at most 3 different values, since 4 or more values would require its domain to be of size at least $1+2+3+4=10>9$. So, there are at most 9 values for the pair $(\pi(x), \pi'(x))$. If there are exactly 9, then each value of $\pi(x)$ (and $\pi'(x)$) must occur exactly 3 times. The partition π therefore cannot have any part of size bigger than 3, hence must have a part of size 3 and 3 parts of size 1. But then the remaining 3 elements must be split into pairs, a contradiction. Hence there are no more than 8 values for the pair $(\pi(x), \pi'(x))$, and by the pigeonhole principle there exist two distinct numbers x and y for which both $\pi(x)=\pi(y)$ and $\pi'(x)=\pi'(y)$.

6. **Answer:** 1. Denote the integral by I. The change of variables $x \mapsto 6-x$ yields:

$$I=\int_2^4 \frac{\sqrt{\ln(x+3)}\,\mathrm{dx}}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}},$$

so

$$2I=\int_2^4 \frac{\sqrt{\ln(9-x)}\,\mathrm{dx}}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}+\int_2^4 \frac{\sqrt{\ln(x+3)}\,\mathrm{dx}}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}=\int_2^4 \mathrm{dx}=2,$$

hence $I=1$.

7. Subtract the nth row from the $(n+1)$st, the $(n-1)$st from the nth, etc. the first from the second. Since $A_{j+1}-A_j=a_j$, we get an upper-triangular matrix with main diagonal $(a_0, a_1, \ldots, a_n)$. Since the determinant of a triangular matrix is the product of its diagonal terms and since the determinant is preserved under elementary row operations, we conclude that the original determinant is equal to $a_0 a_1 a_2 \cdots a_n$.

8. Let a, $b \in N \setminus \{1\}$. Let $x := a-1$, $y := b-1$, $z := 1$. Then x, y, $z \in N$ and

$$xy+xz+yz+1=(a-1)(b-1)+(a-1)+(b-1)+1=ab.$$

9. Since

$$x_n^2-x_{n-1}x_{n+1}=1=x_{n+1}^2-x_n x_{n+2},$$

we get

$$x_n(x_n+x_{n+2})=x_{n+1}(x_{n+1}+x_{n-1}).$$

Dividing this equality by the nonzero number $x_n x_{n+1}$, we see that the value of

$$a_n := \frac{x_{n+1}+x_{n-1}}{x_n}$$

does not depend on n, i.e., is constant. Calling his constant a, we get $x_{n+1}=a x_n - x_{n-1}$.

10. **Answer:** $a=2$, $b=4$. Take logarithms of both sides and divide by $ab \neq 0$. We get $\frac{\ln a}{a}=\frac{\ln b}{b}$. The function $f(x) := \frac{\ln x}{x}$ is (strictly) monotone increasing on the interval $(0, e)$ and (strictly) monotone decreasing on (e, ∞), since $f'(x)=\frac{1-\ln x}{x^2}$ is positive when $\ln x<1$

and negative when $\ln x > 1$. Since $a < b$, the values $f(a)$ and $f(b)$ can coincide only if $a < e < b$. As $a \in \mathbb{N}$, and $f(1) = 0 \neq f(b)$ for any $b \in (e, \infty)$, the number a has to equal 2. A matching value $b = 4$ is found by inspection. By strict monotonicity of f on (e, ∞), this is the only value where $f(b) = f(a)$, so the problem has only one solution $(2, 4)$.

11. **Answer:** $c \in \{\pm\sqrt{1 - \frac{n^2}{4k^2}} : n = 1, 2, \ldots, 2k.\}$ The characteristic equation for this ODE is $r^2 - 2cr + 1 = 0$ and has the roots $c \pm \sqrt{c^2 - 1}$.

Case 1: $|c| > 1$. Let $\omega := \sqrt{c^2 - 1}$. Then the general solution is

$$e^{ct}(Ae^{\omega t} + Be^{\omega t})$$

where A, B are constants. The condition $x(0) = 0$ implies $A = -B$ and then the condition $x(2\pi k) = 0$ implies $B = 0$, since $e^{4\pi k\omega} \neq 0$. So, there are no nontrivial solutions in this case.

Case 2: $|c| = 1$. The general solution is

$$(A + Bt)e^{ct}$$

where A and B are constant. By the condition of the problem, the linear function $A + Bt$ must vanish at two points, 0 and $2\pi k$, so it must be identically zero, hence again there are no nontrivial solutions.

Case 3: $|c| < 1$. Let $\omega := \sqrt{1 - c^2}$. The general solution is

$$e^{ct}(Ae^{i\omega t} + Be^{i\omega t}).$$

The condition $x(0) = 0$ implies $A = -B$ and the condition $x(2\pi k) = 0$ implies that $2\pi k\omega = \pi n$, $n \in \mathbb{Z}$, so

$$c^2 = 1 - \frac{n^2}{4k^2}.$$

The right side is nonnegative and less than 1, hence the possible values for c are $\pm\sqrt{1 - n^2/4k^2}$, $n = 1, 2, \ldots, 2k$.

12. **Answer:** $\binom{n+1}{3}$. Let (i, j) be any pair of numbers from the set $\{1, 2, \ldots, n\}$ (the order of i and j is fixed). There are $n - 1$ ways this pair can be chosen as (a_k, a_{k+1}) and there are $(n - 2)!$ possible permutations of other elements. So, the total contribution from all pairs is

$$(n - 1)(n - 2)! \sum_{i=1}^{n} \sum_{j=1}^{n} (i - j)^2.$$

Dividing by the total number of permutations $n!$, we get the average

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (i^2 - 2ij + j^2) &= \frac{1}{n} \left(2n \sum_{j=1}^{n} j^2 - 2(\sum_{i=1}^{n} i)(\sum_{j=1}^{n} j) \right) \\ &= \frac{1}{n} \left(2n \frac{n(n+1)(2n+1)}{6} - 2 \left(\frac{n(n+1)}{2} \right)^2 \right) \\ &= \frac{n(n+1)}{6} \left(2(2n+1) - 3(n+1)\right) = \binom{n+1}{3}. \end{aligned}$$