Solutions to homework #3.

1. Expanding both sides and simplifying, we get

$$x^2 + xy + xz = yz$$

which is equivalent to $(x+y)(x+z) = 2yz$. But if x, y and z are all odd, then the right-hand side is congruent to 2 mod 4 and the left hand side is congruent to 0 mod 4. So, all three numbers cannot be odd.

2. Of 5 cells in the same column, at least 3 must be of the same color. Let's call a column red if most cells in it are red and blue otherwise. By the pigeonhole principle again, there must be at least 21 red or blue columns, say, red. Next, there are $\binom{5}{3} = 10$ ways to place three red cells among 5 cells, hence there are at least three columns among those 21 where at least 3 red cells appear in the same rows. By choosing these columns and rows, we obtain an entirely red block.

3. Consider the generating function $f(z) := \sum_{n=0}^{\infty} a_n z^n$. The assumption of the problem is equivalent to the condition

$$f^2(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z}.$$

Therefore, $f(z) = \pm\frac{1}{\sqrt{1-z}}$. Since $a_0 = 1$, we must take the plus sign. Thus,

$$f(z) = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-z)^n = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n (n!)^2} z^n.$$

Answer: $a_n = \frac{(2n)!}{2^n (n!)^2}$.

4. First of all, $17 \equiv 7 \pmod{10}$, and Euler's totient function has value 4 at 10: $\phi(10) = 4$. Next, $17 \equiv 1 \pmod 4$, so $17^{17} \equiv 1^{17} \equiv 1 \pmod 4$. Therefore, $17^{17^{17}} \equiv 7^{4k+1} \equiv 7 \pmod{10}$, the last congruence being justified by Euler's theorem. **Answer:** 7.

5. The given integral I can be obtained as the limit of the contour integral $\lim_{R\to\infty} \oint_{\Gamma_R} \frac{dz}{1+z^{2n}}$ where Γ_R is a counterclockwise oriented contour consisting of the segment $[-R, R]$ and the arc $\{Re^{i\theta} : \theta \in [0, \pi]\}$. The contribution from the arc tends to zero, since it does not exceed in absolute value the arc length πR times $1/(R^{2n} - 1)$. By the Residue theorem, the integral over Γ_R is the sum of all residues of the integrand inside Γ_R multiplied by $2\pi i$. The integrand has simple poles at $\omega_k := e^{(1+2k)\pi i/2n} : k = 0, \ldots, n-1$. So,

$$I = 2\pi i \sum_{k=0}^{n-1} \frac{1}{2n\omega_k^{2n-1}} = \frac{\pi i}{n} \frac{1}{\omega_0^{2n-1}} \frac{1 - e^{(2n-1)\pi i}}{1 - e^{(2n-1)\pi i/n}} = \frac{2\pi i}{2ni \sin(\pi/2n)} = \frac{\pi}{n \sin(\pi/2n)}.$$

Answer: $\frac{\pi}{n \sin(\pi/2n)}$.

6. The Taylor coefficients $1/(n+1)!$ of the entire function $(e^z - 1)/z$ are rational and the constant term is equal to 1. The Taylor series for its reciprocal can be obtained by long division of the power series consisting of one constant term 1 by the Taylor series for $(e^z - 1)/z$. Since all operations involved in long division are rational, the resulting coefficients must all be rational numbers.

7. Since the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges everywhere, the series in question converges if and only if so does the series $\sum_{n=0}^{\infty} \frac{n^2}{z^n}$. If $|z| \leq 1$, the nth term of the latter series does not tend to zero, so the series diverges. If $|z| > 1$, then

$$\lim_{\frac{(n+1)^2}{n^2|z|}} = \frac{1}{|z|} < 1,$$

so the series converges by the ratio test.

8. Suppose A has an eigenvalue λ with $|\lambda| > 1$; let v be the associated eigenvector. Since $A^n v = \lambda^n v$, the sequence of powers (A^n) is then unbounded. On the other hand, a product of two row-stochastic matrices (i.e., nonnegative matrices with row sums equal to 1) is again a row-stochastic matrix. Therefore, the sequence (A^n) consists of row-stochastic matrices, hence is bounded. This contradiction proves that A does not have eigenvalues of absolute value greater than 1.

9. By the Binomial theorem,

$$\sum_{U \subseteq S} (-1)^{|U|} \binom{m - \sigma(U)}{|S|}$$

is the coefficient of $x^{|S|}$ in the expression

$$\sum_{U \subseteq S} (-1)^{|U|} (1+x)^{m-\sigma(U)}.$$

Next, $(x+1)^a - 1 = ax + \cdots$ plus higher order terms, so the product $\prod_{a \in S} ((1+x)^a - 1)$ expands to yield $\pi(S) x^{|S|}$ plus higher order terms, we have

$$\sum_{U \subseteq S} (-1)^{|U|} (1+x)^{m-\sigma(U)} = (1+x)^m \sum_{U \subseteq S} \frac{(-1)^{|U|}}{(1+x)^{\sigma(U)}}$$
$$= (1+x)^{m-\sigma(S)} \prod_{a \in S} ((1+x)^a - 1) = (1+x)^{m-\sigma(S)} (\pi(s) x^{|S|} + \cdots).$$

Hence $\sum_{U \subseteq S} (-1)^{|U|} \binom{m-\sigma(U)}{|S|} = \pi(S)$.

10. Let m be the nilpotency index of A (so that $A^m = 0$) and n the nilpotency index of B. Consider $(A+B)^{m+n-1}$. Since A and B commute, the usual Binomial theorem applies to yield

$$(A+B)^{m+n-1} = \sum_{j=0}^{m+n-1} \binom{m+n-1}{j} A^j B^{m+n-1-j}.$$

Now, if $j \geq m$, then $A^j = 0$. If $j < m$, then $m + n - 1 - j > n - 1 \geq n$, and $B^{m+n-1-j} = 0$. Thus, all terms in the binomial expansion of $(A+B)^{m+n-1}$ are zero, hence $(A+B)^{m+n-1} = 0$.

11. First note that the function $x \mapsto \int_0^x f(x-t)e^{-t^2}\,dt$ is continuous whenever f is continuous. Consider the map $T\colon C[0,1] \to C[0,1]$ defined by

$$(Tf)(x) := g(x) + \int_0^x f(x-t)e^{-t^2}\,dt.$$

Given $f,\, h \in C[0,1]$, we have

$$\begin{aligned} \|Tf - Th\|_\infty &\le \sup_{x\in[0,1]} \int_0^x |f(x-t) - h(x-t)|e^{-t^2}\,dt \\ &\le \|f-h\|_\infty \sup_{x\in[0,1]} \int_0^x e^{-t^2}\,dt = \|f-h\|_\infty \int_0^1 e^{-t^2}\,dt. \end{aligned}$$

Since $0 < \int_0^1 e^{-t^2}\,dt < 1$, the map T is a proper contraction. Since $C[0,1]$ is a complete metric space, by the Banach fixed point theorem, there is a (unique) function $f \in C[0,1]$ such that $Tf = f$, as desired.

12. Note that

$$p(j) - 1 = \frac{2}{\sqrt{3}}\mathrm{Im}(e^{\pi \mathrm{i}/3}\omega^j), \qquad j = 0, \ldots, 3n, \qquad \omega := e^{2\pi \mathrm{i}/3}.$$

Hence

$$p(x) = 1 + \frac{2}{\sqrt{3}} \sum_{k=0}^{3n} \mathrm{Im}(e^{\pi \mathrm{i}/3}(\omega - 1)^k).$$

Hence

$$p(3n+1) = 1 + \frac{2}{\sqrt{3}}\mathrm{Im}\left(e^{\pi \mathrm{i}/3}(\omega^{3n+1} - (\omega-1)^{3n+1})\right) = -\frac{2}{\sqrt{3}}\mathrm{Im}(e^{\pi \mathrm{i}/3}(\omega-1)^{3n+1}).$$

Since $\omega - 1 = \mathrm{i}\sqrt{3}e^{\pi \mathrm{i}/3}$, an application of de Moivre's theorem yields

$$p(3n+1) = 1 + 2(\sqrt{3})^{3n} \sin\left((3n+1)\pi/6\right).$$

We are given that $p(3n+1) = 720 = 1 + 3^6$, so the only matching value of n is $n = 4$. **Answer: 4.**