Solutions to homework #4.

1. Let p denote the unknown polynomial. Then $p(x) = (x^2 - 1)q(x) + r(x)$, with q the quotient and r the remainder from the division of $p(x)$ by $x^2 - 1$. The polynomial r is linear and, by the assumption of the problem, the line $y = r(x)$ passes through the points $(-1, -5)$ and $(1, 5)$. Hence that line is $r(x) = 5x$.

2. Since ABC is a right triangle, $AEDF$ is a rectangle. The diagonals of a rectangle are equal, hence we need to minimize AD, which is achieved by making AD perpendicular to the line BC. So, D must be at the foot of the perpendicular to BC from the vertex A.

3. Consider the sequence $a_n := (1 + \sqrt{3})^{2n+1} + (1 - \sqrt{3})^{2n+1}$. Since both geometric series $((1 \pm \sqrt{3})^2)^n$ satisfy the recurrence whose characteristic polynomial is $\lambda^2 - 8\lambda - 4 = 0$, the sequence (a_n) satisfies the recurrence relation $a_{n+1} - 8a_n - 4a_{n-1} = 0$. Since $a_0 = 2$, $a_1 = 20$ and the recurrence satisfied by (a_n) has integer coefficients, each a_n is an integer. Since $-1 < 1 - \sqrt{3} < 0$, we conclude that $\lfloor (1 + \sqrt{3})^{2n+1} \rfloor = a_n$.

Prove by induction that 2^{n+1} is the highest power dividing a_n. The base cases $n = 0, 1$ are checked directly, and

$$a_{n+1} = 2^{n+2} \left(2\frac{a_n}{2^n} + \frac{a_{n-1}}{2^n} \right).$$

By the inductive hypothesis, the numbers $a_n/2^n$, $a^{n-1}/2^n$ are integers, the former even and the latter odd. Hence the expression in parentheses is an odd integer, which finishes the proof.

4. Let

$$g(z) = \sum_{j=0}^{2n} a_j z^j = a_{2n} \prod_{j=1}^{2n} (z - z_j).$$

Then

$$z^{2n}\overline{g}(1/z) = \sum_{j=0}^{2n} \overline{a_j} z^{2n-j} = \overline{a_{2n}} \prod_{j=1}^{2n} (1 - \overline{z_j} z).$$

In particular, we see that $a_0 = \overline{a_{2n}} \neq 0$. So, 0 is not a root of g and, for every root z_j, $1/\overline{z_j}$ is also a root. So, the multiset of roots (with roots listed according to their multiplicities) consists of pairs of the form $\{z_j, 1/\overline{z_j}\}$ with $|z_j| \neq 1$ and an even number of points on the unit circle, for which $z_j = 1/\overline{z_j}$.

Conversely, any such multiset of size $2n$ gives rise to a polynomial satisfying the assumption of the problem. Indeed, the leading coefficient can always be chosen so that $a_{2n} = \overline{a_{2n}} \prod_{j=1}^{2n} (-\overline{z_j})$ and then $g(z) = z^{2n}\overline{g}(1/z)$.

5. Let $x^3 + px + qx + r$ be the monic polynomial with roots a, b, c. Then $p = -(a+b+c) = 0$, $q = ab + bc + ca$ and $r = -abc$. Each of the numbers a, b and c satisfies $x^5 + qx^3 + rx^2 = 0$, so $a^5 + b^5 + c^5 = -q(a^3 + b^3 + c^3) - r(a^2 + b^2 + c^2)$. Next, the identity $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ac - bc - ca)$ implies that $a^3 + b^3 + c^3 = 3abc = -3r$ and the identity $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$ implies $a^2 + b^2 + c^2 = -2q$. Therefore,

$a^5+b^5+c^5 = -q(-3r)-r(-2q) = 5qr$ and $\frac{a^3+b^3+c^3}{3}\ \frac{a^2+b^2+c^2}{2} = (-r)(-q) = qr$, which finishes the proof.

6. Consider the function $f(x) = 1/x$ on the interval $[1, 2]$. Partition the interval using the points $\{1, 1+\frac{1}{n}, 1+\frac{2}{n}, \dots, 2\}$ and form the Riemann sum with the right endpoint of each interval. We get

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} f\left(1+\frac{j+1}{n}\right)\frac{1}{n} = \lim_{n\to\infty} \sum_{j=0}^{n-1} \frac{1}{n+j+1} = \int_1^2 \frac{\mathrm{d}x}{x} = \ln 2.$$

7. It is enough to prove that, for every $1 > a > 0$, the function

$$g_a(z) := g(z) := \frac{1}{4a^2}\int_{-a}^{a}\int_{-a}^{a} f(z+x+\mathrm{i}y)\,\mathrm{d}x\,\mathrm{d}y$$

is constant for all $z \in \mathbb{C} = \mathbb{R}^2$, since then we get the constancy of f for $a \to \infty$. Note that g also satisfies the assumption of the problem, namely it takes on values in the interval $[0, 1]$ and has the mean value property. Indeed, by Fubini's theorem,

$$\frac{1}{2\pi}\int_0^{2\pi} g(z+e^{\mathrm{i}t})\,\mathrm{d}t = \frac{1}{8\pi a^2}\int_0^{2\pi}\int_{-a}^{a}\int_{-a}^{a} f(z+e^{\mathrm{i}t}+x+\mathrm{i}y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$= \frac{1}{4a^2}\int_{-a}^{a}\int_{-a}^{a}\frac{1}{2\pi}\int_0^{2\pi} f(z+e^{\mathrm{i}t}+x+\mathrm{i}y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \frac{1}{4a^2}\int_{-a}^{a}\int_{-a}^{a} f(z+x+\mathrm{i}y)\,\mathrm{d}x\,\mathrm{d}y = g(z).$$

The function g is also uniformly continuous, hence satisfies

$$|g(z) - g(z') \le \delta(|z-z'|)(|z-z'|) \quad \text{where} \quad \lim_{\epsilon\to 0}\delta(\varepsilon) = 0. \tag{1}$$

Now let $\mathcal{G}$ be the set of all functions $g : \mathbb{R}^2 \to [0, 1]$ that satisfy the mean value property and the smoothness property (1) with a fixed function δ. The set $\mathcal{G}$ consists of uniformly bounded and uniformly equicontinuous functions, hence $\mathcal{G}$ is compact with respect to the uniform norm. Hence the functional $g \mapsto g(1) - g(0)$ attains its supremum α on $\mathcal{G}$ for some g_{a^*}. Since $\mathcal{G}$ is rotation and translation invariant, we can conclude that

$$|g(z) - g(z')| \le \alpha \quad \text{for all } g \in \mathcal{G} \quad \text{whenever } |z-z'| = 1. \tag{2}$$

But then, by the mean value property,

$$\alpha = g_{a^*}(1) - g_{a^*}(0) = \frac{1}{2\pi}\int_0^{2\pi}(g_{a^*}(1+e^{\mathrm{i}t}) - g_{a^*}(e^{\mathrm{i}t}))\,\mathrm{d}t \le \frac{1}{2\pi}\int_0^{2\pi}\alpha\,\mathrm{d}t = \alpha.$$

This is possible only with equality under the integral sign, i.e.,

$$g_{a^*}(1+e^{\mathrm{i}t}) - g_{a^*}(e^{\mathrm{i}t}) = \alpha \quad \text{for all } t.$$

In particular, $g_{a^*}(2) - g_{a^*}(1) = \alpha$. But then $g_{a^*}(n) - g_{a^*}(1) = n\alpha$. Since g is bounded by 1, we conclude that $\alpha = 0$. Now (2) implies that $g(z) = g(z')$ for all $g \in \mathcal{G}$ whenever $|z-z'| = 1$. Since any two points on the plane can be joined by a path with consecutive points at distance 1, we conclude that every function in $\mathcal{G}$ is constant. Therefore f is constant as well.

8. Suppose the function f is not monotonic. Then there are points $x_1 < x_2 < x_3$ such that $f(x_1) \le f(x_2)$, $f(x_2) \ge f(x_3)$ (or $f(x_1) \ge f(x_2)$, $f(x_2) \le f(x_3)$, which reduces to the first case by flipping the sign of f). By continuity of f and compactness of $[x_1, x_3]$, f attains its maximum on $[x_1, x_3]$. Since the value of f at x_2 is no less than the values at endpoints, there is at least one interior point $x_* \in (x_1, x_3)$ where the maximum is attained. But then an open sufficiently small neighborhood of x_* is mapped to a half-open interval with right endpoint $f(x_*)$. Contradiction! So, f is monotonic.

9. The generating function p_1 for partitions where each part is repeated at most 3 times is

$$p_1(x) = \prod_{k=1}^{\infty}(1 + x^k + x^{2k} + x^{3k}) = \prod_{k=1}^{\infty}(1 + x^k)(1 + x^{2k}).$$

The generating function p_2 for partitions where all even parts are distinct is the product of the generating function for odd partitions $\prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}}$ and the generating function for even partitions with distinct parts $\prod_{k=1}^{\infty}(1 + x^{2k})$. But then

$$\begin{aligned} p_2(x) &= \prod_{k=1}^{\infty} \frac{1}{1 - x^{2k-1}} \prod_{k=1}^{\infty}(1 + x^{2k}) = \prod_{k=1}^{\infty} \frac{1}{1 - x^{2k-1}} \cdot \frac{1 - x^{4k}}{1 - x^{2k}} = \prod_{k=1}^{\infty} \frac{1}{1 - x^k} \prod_{k=1}^{\infty}(1 - x^{4k}) \\ &= \prod_{k=1}^{\infty} \frac{1}{1 - x^k} \prod_{k=1}^{\infty}(1 - x^k)(1 + x^k)(1 + x^{2k}) = \prod_{k=1}^{\infty}(1 + x^k)(1 + x^{2k}) = p_1(x). \end{aligned}$$

10. For a fixed difference d, the possible arithmetic progressions are $\{1, 1 + d, 1 + 2d, 1 + 3d, 1 + 4d\}$ through $\{n - 4d, n - 3d, n - 2d, n - d, n\}$. So, the number of arithmetic progressions with difference d is $n - 4d$. Since $n - 4d \ge 1$, we have $1 \le d \le \frac{1}{4}(n - 1)$, so the number of all possible 5-term progressions is

$$\sum_{d=1}^{\lfloor \frac{1}{4}(n-1) \rfloor} (n - 4d).$$

given a 5-term arithmetic progression $\{1, 1 + d, 1 + 2d, 1 + 3d, 1 + 4d\}$, the possible 3-term progressions are $\{a, a + d, a + 2d\}$, $\{a + d, a + 2d, a + 3d\}$, $\{a + 2d, a + 3d, a + 4d\}$, and $\{a, a + 2d, a + 4d\}$, so there are $4 \cdot 3! \cdot 2! = 48$ ways to arrange such 5 numbers so that the first three, when ordered, form an arithmetic progression. Hence the probability of obtaining such a draw is

$$\begin{aligned} \frac{48}{n(n-1)(n-2)(n-3)(n-4)} \sum_{d=1}^{\lfloor \frac{1}{4}(n-1) \rfloor} (n - 4d) &\ge \frac{48(n\lfloor \frac{n-1}{4} \rfloor - 4 \cdot \frac{1}{2}(1 + \lfloor \frac{n-1}{4} \rfloor)\lfloor \frac{n-1}{4} \rfloor)}{n(n-1)(n-2)(n-3)(n-4)} \\ &= \frac{48\lfloor \frac{n-1}{4} \rfloor(n - 2 - 2\lfloor \frac{n-1}{4} \rfloor)}{n(n-1)(n-2)(n-3)(n-4)} \ge \frac{48(\frac{n-1}{4} - \frac{3}{4})(n - 2 - 2\frac{n-1}{4})}{n(n-1)(n-2)(n-3)(n-4)} \\ &= \frac{6(n-4)(n-3)}{n(n-1)(n-2)(n-3)(n-4)} = \frac{6}{n(n-1)(n-2)} \ge \frac{6}{(n-2)^3}. \end{aligned}$$

11. Consider the function $f(x) := \sqrt{px}$. The number of lattice points under the curve

$y = f(x)$ with x-coordinates $1 \le x \le n$ where $p = 4n+1$ is

$$\sum_{x=1}^{n} \lfloor f(x) \rfloor = \lfloor \sqrt{p} \rfloor + \lfloor \sqrt{2p} \rfloor + \cdots + \lfloor \sqrt{p(p-1)/4} \rfloor .$$

Alternatively, this number of lattice points can be calculated by reflecting the region about the line $x = y$, i.e., by working with the inverse function $g(x) = x^2/p$. The largest integer smaller than $\sqrt{np}$ is $(p-1)/2 < \sqrt{(p^2-p)/2}$, since the next integer $(p+1)/2$ is already greater than $\sqrt{np}$. So, the number of lattice points under the curve $y = g(x)$ with x-coordinates $1 \le x \le (p-1)/2$ is

$$\begin{aligned}\sum_{x=1}^{(p-1)/2} \lfloor g(x) \rfloor &= \sum_{x=1}^{(p-1)/2} \left\lfloor \frac{x^2}{p} \right\rfloor = \sum_{x=1}^{(p-1)/2} \left(\frac{x^2}{p} - \left\{ \frac{x^2}{p} \right\} \right) = \frac{1}{p}(1^2 + 2^2 + \cdots + \left(\frac{p-1}{2}\right)^2) \\ &- \sum_{x=1}^{(p-1)/2} \left\{ \frac{x^2}{p} \right\} = \frac{p^2-1}{24} - \sum_{x=1}^{(p-1)/2} \left\{ \frac{x^2}{p} \right\}.\end{aligned}$$

Note that $r_x := p\left(\left\{\frac{x^2}{p}\right\}\right) \equiv x^2 \pmod p$. Since there are $(p-1)/2$ quadratic residues mod p and each is congruent to exactly one r_x, the sequence $(r_1, \ldots, r_{(p-1)/2})$ lists all the quadratic residues mod p exactly once. But $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = 1$, so -1 is a quadratic residue mod p. Then r is a quadratic residue if and only if so is $-r$. Therefore the set $\{r_1, \ldots, r_{(p-1)/2}\}$ can be partitioned into n pairs (r_i, r_j) so that, for each pair, $r_i + r_j = p$. Thus $\sum_{x=1}^{(p-1)/2} r_x = np$. Hence

$$\sum_{x=1}^{(p-1)/2} \lfloor g(x) \rfloor = \frac{p^2-1}{24} - \frac{np}{p} = \frac{p^2-1}{24} - \frac{p-1}{4}.$$

The number of lattice points under $y = f(x)$ is equal to the number of lattice points above $y = g(x)$, i.e., is the difference between the number $n \times \sqrt{2p}$ of points in the rectangle and the number we just obtained. Thus,

$$\lfloor \sqrt{p} \rfloor + \lfloor \sqrt{2p} \rfloor + \cdots + \lfloor p(p-1)/4 \rfloor = \frac{(p-1)^2}{8} - \left(\frac{p^2-1}{24} - \frac{p-1}{4} \right) = \frac{p^2-1}{12}.$$

12. Let $\langle \cdot, \cdot \rangle$ denote the scalar product on H. Define a new scalar product by

$$\langle x, y \rangle_{\text{new}} := \sum_{j=0}^{n-1} \langle T^j x, T^j y \rangle .$$

Since

$$\langle x, x \rangle \le \langle x, x \rangle_{\text{new}} \le \left(\sum_{j=0}^{n-1} \|T^j\|^2 \right) \langle x, x \rangle \qquad \text{for all } x \in H,$$

the norm induced by $\langle \cdot, \cdot \rangle_{\text{new}}$ is equivalent to the norm induced by $\langle \cdot, \cdot \rangle$. Therefore, there is a bounded and boundedly invertible operator $A : H \to H$ such that $\langle Ay, Ax \rangle = \langle y, x \rangle_{\text{new}}$.

Then, for $y := A^{-1}x$, we have

$$\begin{aligned}\langle x, x\rangle - \langle ATA^{-1}x, ATA^{-1}x\rangle &= \langle Ay, Ay\rangle - \langle ATy, ATy\rangle = \langle y, y\rangle_{\text{new}} - \langle Ty, Ty\rangle_{\text{new}} \\ &= \sum_{j=0}^{n-1}\langle T^j y, T^j y\rangle - \sum_{j=0}^{n-1}\langle T^{j+1}y, T^{j+1}y\rangle = \langle y, y\rangle - \langle T^n y, T^n y\rangle,\end{aligned}$$

which is nonnegative sincse $\|T^n\| \leq 1$. Thus,

$$\langle x, x\rangle \geq \langle ATA^{-1}x, ATA^{-1}x\rangle \qquad \text{for all } \ x \in H,$$

hence $\|ATA^{-1}\| \leq 1$.