Solutions to homework #5.

1. Starting with a_1 and iterating the given inequality, we obtain

$$0 < a_1 \leq \sum_{j=2^n}^{2^{n+1}-1} a_j \quad \text{for all } \ n \in \mathbb{N}.$$

Hence the tail of the series $\sum_{n=1}^{\infty} a_n$ does not tend to zero, so the series is divergent.

2. Note that the numerator is equal to the difference of squares, precisely,

$$(x+1/x)^6 - (x^6+1/x^6) - 2 = \Big((x+1/x)^3 - (x^3+1/x^3)\Big)\Big((x+1/x)^3 + (x^3+1/x^3)\Big).$$

Hence the whole fraction is simply $(x+1/x)^3 - (x^3+1/x^3) = 3(x+1/x)$, and the minimum of the obtained expression is well-known to be 6.

3. From the congruence

$$2^m \equiv -1 \pmod{p}$$

we get

$$2^{2^k m} \equiv 1, \qquad 2^{2^k m} + 1 \equiv 2 \pmod{p}$$

for any m, $k \in \mathbb{N}$ and an odd prime p. This implies that

$$(2^{2^k}+1, 2^{2^l}+1) = 1 \quad \text{for all } \ k, l \in \mathbb{N}, \ k \neq l.$$

Note that the more general statement $(2^m+1, 2^n+1) = 1$ for $m \neq n$ is false: for example, 3 and 33 are *not* relatively prime.

4. For simplicity, let's say positions of the wheels correspond to triples of numbers from the set $\{1, \ldots, 8\}$. First observe that the correct combination (a, b, c) contains three numbers, so by the pigeonhole principle

- some two of the numbers come from $\{1, 2, 3, 4\}$, or
- some two of the numbers come from $\{5, 6, 7, 8\}$.

Consider any Latin square made up of numbers 1 through 4, e.g.,

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{matrix}.$$

Now convert each ordered pair (x, y) with x, $y \in \{1, 2, 3, 4\}$ into an ordered triple (x, y, z) by taking z to be the element of the Latin square in row x and column y. This gives 16 tries that will open the lock if two numbers from the correct combination belong to the set $\{1, 2, 3, 4\}$. Otherwise use the analogous scheme for testing the remaining numbers 5 through 8. Therefore, the safe can be opened in 32 or fewer tries.

To see that 32 is the minimum, we need to show that, no matter what 31 attempts have been made, there is some combination (a, b, c) that has been missed completely, i.e., none of the triples of the form (a, b, x), (x, b, c), (a, x, c) has been tested, where x is arbitrary. To this end, let us keep track of the attempts made in an 8×8 array R by entering the number z of the trial (x, y, z) in the cell with coordinates (x, y). If we make several trials with the same two coordinates (x, y), then all corresponding numbers z will be entered in that one cell. One of four cases must occur.

(i) Suppose some row or column contains exactly two entries; for definiteness, suppose row 3 contains only 4 and 7. Then there are at least 6 columns which do not have an entry in row 3. Now, each of these 6 columns could not have as many as 5 entries, for that would require 30 entries and there are only 29 other trials recorded in R. It follows that one of these columns must hold no more than 4 entries. Altogether the entries in row 3 and the chosen column (say column 7) account for only 6 of the 8 possible numbers. Taking any number j that does not occur, we see that the triple $(3, 7, j)$ has been missed completely.

This argument goes through with equal force in cases when a row or column contains exactly one entry or no entry at all. Since 31 trials leave some row with no more than 3 entries, it remains to consider the case of a row that contains exactly 3 entries.

(ii) Suppose row 1 contains 3 entries in columns 1, 2, 3. If any of these columns were to contain exactly one or exactly two entries, our claim would follow as in the previous cases. Suppose then that each of these columns has at least 3 entries, for a total of at least 9 in these columns altogether. The other 5 columns then cannot each have as many as 5 entries, for there are at most 22 trials recorded in them. Thus one of these other columns should have 4 entries or fewer and our first argument applies.

If 2 or more of the 3 entries in row 1 occur in the same cell, our argument is only reinforced, since three entries in the same cell lead to not more than 28 in the remaining 7 columns, and 3 entries in 2 cells imply at least 6 in their 2 columns, so leave not more than 25 entries in the remaining 6 columns. At any rate, one of the other columns must have no more than 4 entries, and this column and row 1 do not contain all 8 numbers.

Thus, there is at least one combination that remains completely untested if the number of trials is below 32.

5. The proof is by contradiction. Suppose that the integers 1 through N are colored red, green and blue so that each color is given to more than $N/4$ integers and there are no x, y, z of distinct colors such that $x = y + z$. Assume that 1 is red. Then there are no green and blue integers whose difference is 1. We will call a nonempty set $S \subseteq \{1, \ldots, N\}$ an *interval* if it consists of consecutive integers. If we delete the red integers, the remaining ones can be partitioned into intervals. Moreover, each interval has to be monochromatic by the observation above.

Suppose there exist intervals containing at least 2 integers of both green and blue colors. Let A and B denote the longest green and blue interval, respectively. If $a \in A$ and $b \in B$, then $|a - b|$ is not red. Hence the set $C := \{|a - b| : a \in A, b \in B\}$ does not contain any red integer. If $A = [a_1, a_2]$, $B = [b_1, b_2]$, then

$$C = \begin{cases} [b_1 - a_2, b_2 - a_1] & \text{if } a_2 < b_1 \\ [a_1 - b_2, a_2 - b_1] & \text{if } b_2 < a_1, \end{cases}$$

so it is an interval containing more integers than either A or B. But this is a contradiction to the choice of A and B.

On the other hand, at least one of the colors green or blue is assigned to two consecutive integers, since otherwise the number of red ones would be greater than or equal to the combined number of green and blue integers. In that case, the combined number of blue and green integers would be at most $N/2$, which contradicts the assumption that each color is given to more than $N/4$ integers.

Suppose now that there are no consecutive green integers but there are two consecutive blue integers, i.e., $|B| \geq 2$ for the longest blue interval B. If $1 \leq s \leq N$ is green, then $s \geq 2$ and $s-1$ and $s+1$ are red. Suppose that the distance between any two green integers is at least 3. Then the intervals $[s-1, s+1]$ are pairwise disjoint for the green integers s and each of them contains two read integers. If N is not green, then this implies that the number of red integers is at least twice the number of the green integers, which is impossible if each color is given to more than $N/4$ integers. If N is green, then the number $n_r \geq 2n_g - 1$ where n_g (n_r) is the number of green (red) integers. If 2 is not green, then 1 is not included in the number of red integers counted so far, so $n_r \geq 2n_g$, a contradiction as before. If 2 is green, take a blue interval $B = [b_1, b_2]$ such that $|B| \geq 2$, $b_2 < N$. Now, $b_2 - 1$ is blue, $b_2 + 1$ is red, and 2 is green. This triple is then a solution to $x = y + z$ in distinct colors, a contradiction.

So we may assume that there is a green integer s such that $s+2$ is green as well. Either $b_1 > s+2$ or $b_2 < s$, since $|B| \geq 2$ and $B \cap \{s, s+2\} = \emptyset$. We may assume $b_1 > s+2$. Consider the set

$$C := \{b - s : b \in B\} \cup \{b - s - 2 : b \in B\}.$$

Since $b_1 < b_2$, C is an interval such that $|C| = |B| + 2$. Now, C does not contain any red integers, so is monochromatic. But C is not green because $|C| > 2$ and not blue because of the maximal choice of B. This last contradiction finishes the proof.

6. The sesquilinear form on $n \times n$-matrices

$$\langle A, B \rangle := \operatorname{tr}(AB^*)$$

is easily checked to satisfy all the axioms of an inner product, hence it satisfies the Cauchy-Schwarz inequality, which is exactly what we are asked to prove.

7. This integral can be evaluated in a variety of ways. Here is a brute force approach. Consider the function

$$f(z) = \frac{e^{\mathrm{i}z}}{z(z-\pi)}$$

and the contour $\Gamma_R \cup [-R, -\varepsilon] \cup \gamma_0 \cup [\varepsilon, \pi - \varepsilon] \cup \gamma_\pi \cup [\pi + \varepsilon, R]$, where Γ_R is the half-circle $\{Re^{\mathrm{i}\phi} : \phi \in [0, \pi]\}$, and γ_0 and γ_π are similar half-circles of radius ε with centers 0 and π, respectively, drawn upwards. Traverse the contour counterclockwise. The function f has no singularities inside the contour, hence the integral along the contour is zero. On the other hand,

$$\lim_{\varepsilon \to 0} \int_{\gamma_0} f(z)\,\mathrm{d}z = -\pi\mathrm{i}\operatorname{Res}(f(z), 0) = \mathrm{i}, \quad \lim_{\varepsilon \to 0} \int_{\gamma_\pi} f(z)\,\mathrm{d}z = -\pi\mathrm{i}\operatorname{Res}(f(z), \pi) = \mathrm{i},$$

$$\lim_{R\to\infty}\int_{\Gamma_R} f(z)\,\mathrm{d}z = 0.$$

Since our integral is simply the imaginary part of the limit of the integral along two segments included in the contour, we get

$$\int_{-\infty}^{\infty} \frac{\sin x\,\mathrm{d}x}{x(x-\pi)} = -2.$$

8. Consider the function

$$g(x) := \det \begin{bmatrix} r_1+x & a+x & a+x & \cdots & a+x \\ b+x & r_2+x & a+x & \cdots & a+x \\ b+x & b+x & r_3+x & \cdots & a+x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b+x & b+x & b+x & \cdots & r_n+x \end{bmatrix}.$$

It can be checked directly that g is a linear function (by elementary row operations or, equivalently, by noticing that the new matrix is a rank-one modification of the given constant matrix). We know that $g(-a) = f(a)$ and $g(-b) = f(b)$. Linearly interpolating between those two values, we get

$$g(0) = \frac{af(b) - bf(a)}{a-b},$$

which is exactly what we wanted to determine.

9. Let P be the set of all linear combinations of the given functions. By Weierstraß's theorem, the set of polynomials is dense in $C[0,1]$, so it is enough to prove that every power x^n, $n = 0, 1, 2\ldots$ can be uniformly approximated by polynomials from P. For $n = 0, 1$, $x^n \in P$. For $n > 1$, consider

$$f_m^n(x) := \sum_{j=0}^{m-1} (-1)^j \frac{m-j}{m}\left(x^{n^{2^j}} + x^{n^{2^{j+1}}}\right) = x^m + \frac{1}{m}\sum_{j=1}^{m}(-1)^{j-1} x^{n^{2^j}} =: x^m + \frac{1}{m}A(x).$$

The sum that defines $A(x)$ consists of terms that alternate in sign and decrease in absolute value, and the first term lies between 0 and 1. Hence $A(x)$ also lies between 0 and 1. Therefore

$$|x^n - f_m^n(x)| \le \frac{1}{m} \quad \text{for all } \ x \in [0,1].$$

10. First of all, it is enough to show that the function

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} =: f_n(x)$$

does not have two consecutive negative zeros. To prove this claim by contradiction, suppose we could have

$$f_n(a) = f_n'(a) + \tfrac{a^n}{n!} = 0, \qquad f_n(b) = f_n'(b) + \tfrac{b^n}{n!} = 0,$$
$$\mathrm{sign} f_n'(a) = \mathrm{sign} f_n'(b) \neq 0. \tag{1}$$

By (1), f_n' would then have an even number of zeroes, but by Rolle's theorem f_n' has an odd number of zeroes between two consecutive roots of f_n. Contradiction!

11. By the assumption of the problem, the Fourier series for the function f on the interval $[0, 2\pi]$ may be differentiated term by term to produce a series for f'. Moreover, the zeroth coefficient in the Fourier series for f is zero, since $\int_0^{2\pi} f(x)\,\mathrm{d}x = 0$. So,

$$f(x) = \sum_{n\in\mathbb{Z},\, n\neq 0} c_n e^{\mathrm{i}nx}.$$

Differentiating term by term, we get

$$f'(x) = \sum_{n\in\mathbb{Z},\, n\neq 0} \mathrm{i}nc_n e^{\mathrm{i}nx}.$$

Now, both f and f' are real-valued, so $f^2 = |f|^2$ and $(f')^2 = |f'|^2$, so by the Parseval formula the left-hand side is equal to

$$2\pi \sum_{n\in\mathbb{Z},\, n\neq 0} |c_n|^2,$$

and the right-hand side is equal to

$$2\pi \sum_{n\in\mathbb{Z},\, n\neq 0} |\mathrm{i}nc_n|^2 = 2\pi \sum_{n\in\mathbb{Z},\, n\neq 0} n^2|c_n|^2,$$

which is indeed larger than the left-hand side.

12. We will prove that a field F with the desired properties exists if and only if x is either transcendental or algebraic and one of its conjugates is nonreal. First, if x belongs to one of these classes, then there exists an automorphism ϕ of $\mathbb{C}$ for which $x \notin \phi(\mathbb{R})$. Then $F := \phi(\mathbb{R})$ will be the suitable proper subfield of $\mathbb{C}$.

Conversely, suppose that x is algebraic and all conjugates of x are real. Suppose $\mathbb{C} = L(x)$ for a proper subfield L of $\mathbb{C}$. Then x is also algebraic over L, so $\mathbb{C}$ is a finite extension of L. Now use the following well-known fact.

Theorem. *If K is an algebraically closed field,* $\mathrm{char}K = 0$*, and K is a finite extension of some proper subfield L, then $|K : L| = 2$.*

This theorem implies that $|\mathbb{C} : L| = 2$. Now let us prove that $\mathrm{i} \notin L$. Suppose $c, d \in L$, $\mathrm{i} \in L$, and $x^2 + cx + d = 0$. Then $x = c' + \sqrt{d'}$ for some $c', d' \in L$, hence $\mathbb{C} = L(\sqrt{d'})$. But this implies $\sqrt[4]{d'} = a + b\sqrt{d'}$, with suitable $a, b \in L$. Thus $\sqrt{d'} = a^2 + b^2d' + 2ab\sqrt{d'}$, hence $a^2 + b^2d' = 0$ and $2ab = 1$. Therefore $4a^4 + 4a^2b^2d' = 0$ and $4a^2b^2 = 1$. This gives $4a^4 + d' = 0$, $d' = -4a^4$, so $\sqrt{d'} = \pm\mathrm{i}2a^2$, showing that $\sqrt{d'} \in L$, a contradiction to $L \neq \mathbb{C}$.

So, $\mathrm{i} \notin L$, hence $\mathbb{C} = L(\mathrm{i})$. This implies that L is a real closed field, hence can be ordered. Now consider the subfield $A := \{y \in L : y \text{ is algebraic}\}$. Then A can be ordered too. It is known that every ordering of every subfield of the field of all algebraic numbers is Archimedean, so A is a field with an Archimedean ordering, hence there exists an embedding $\phi : A \to \mathbb{R}$. But then there exists an extension ϕ' of ϕ such that ϕ' is an embedding of all algebraic numbers to $\mathbb{C}$. Now $L(\mathrm{i}) = \mathbb{C}$ implies that $A(\mathrm{i})$ is the set of all algebraic numbers. Also, $\phi'(\mathrm{i})$ is necessarily $\pm\mathrm{i}$. Since ϕ' permutes the conjugates of x among themselves, $\phi'(x) \in \mathbb{R}$. So, $\phi'(x) \in \phi'(A(\mathrm{i}))$, so $\phi'(x) \in \phi'(A(\mathrm{i})) \cap \mathbb{R} = \phi'(A)$. Thus $\phi'(x) \in \phi'(A)$, that is, $x \in A$, which is a contradiction we wanted to obtain.