

Solutions to homework #6.

1. Since $i^i = e^{i \ln i}$ and $\ln i = \ln |i| + i \arg i = i(\pi/2 + 2\pi k)$, $k \in \mathbb{Z}$, the values of i^i are $\{e^{-(\pi/2+2k\pi)} : k \in \mathbb{Z}\}$.

2. **Answer:** an integer N is within 250 of exactly 15 perfect squares if and only if either $315 \leq N \leq 325$ or $332 \leq N \leq 350$.

Solution. The squares within 250 of N must be consecutive, i.e., form a sequence of the form

$$m^2, (m+1)^2, \dots, (m+14)^2 \tag{1}$$

for some $m \geq 0$. If $m = 0$, then $14^2 \leq N + 250 \leq 15^2$, contradicting $N > 0$.

Now, given N , $m > 0$, the following two conditions are necessary and sufficient for (1) to be the squares within 250 of N :

$$\begin{aligned} (m+14)^2 \leq N + 250 \leq (m+15)^2 - 1 \\ m^2 \geq N - 250 \geq (m-1)^2 + 1. \end{aligned}$$

Subtracting those two, we get $28m + 196 \leq 500 \leq 32m + 222$, which implies $m = 9$ or 10 . If $m = 9$, we get $315 \leq N \leq 325$. If $m = 10$, then $332 \leq N \leq 350$.

3. If two participants, A and B , do not have a common language, then in any triple (A, B, C) it must be C who has a language in common with either A or B (or both). Now, by assumption there are not more than 198 others who speak a particular language spoken by A or B , and since A and B don't know more than 10 languages between them, there cannot be more than $10 \cdot 198 = 1980$ people C who can form an admissible triple (A, B, C) . This falls short of $1985 - 2 = 1983$ others who are present.

Now, if every pair (A, B) has a common language, then each of the other 1984 people must speak one of five or fewer languages spoken by A , or an average of at least 397 (counting A).

4. Let O_1 and O_2 be the centers of C_1 and C_2 , respectively, (We assume that C_1 has radius 1 and C_2 has radius 3.) Then the locus in question is an annulus centered at the midpoint O of O_1O_2 , with inner radius 1 and outer radius 2.

Indeed, if X lies on C_1 and Y lies on C_2 , and M is the midpoint of XY , then

$$\vec{OM} = \frac{1}{2} (\vec{OX} + \vec{OY}) = \frac{1}{2} (\vec{OO}_1 + \vec{O}_1\vec{X} + \vec{OO}_2 + \vec{O}_2\vec{Y}) = \frac{1}{2} (\vec{O}_1\vec{X} + \vec{O}_2\vec{Y}).$$

By the triangle inequality, the length of $\vec{O}_1\vec{X} + \vec{O}_2\vec{Y}$ lies between $3 - 1$ and $3 + 1$. Conversely, every vector of length l between $3 - 1$ and $3 + 1$ can be expressed as the sum of a vector of length 1 and a vector of length 3 by building a (possible degenerate) triangle of side lengths $l, 1, 3$. Thus the set of points M is exactly the closed annulus with center at O , inner radius 1 and outer radius 2.

5. Since the left-hand side is bigger than 3, the value of z must be at least 2, so

$$5^x 7^y + 4 \equiv (-1)^x \cdot 1^y + 1 \equiv 0 \pmod{3},$$

implying that x is odd. Hence x is nonzero, and considering the equation modulo 5, we get

$$5^x 7^y + 4 \equiv 4 \equiv 3^z \pmod{5}.$$

Now, the powers of 3 give the following sequence of residues mod 5:

$$3, 4, 2, 1, 3, 4, 2, 1, \dots$$

So, to satisfy the condition $3^z \equiv 4 \pmod{5}$, z must be $2 \pmod{4}$. For $z = 2$, we have a solution $x = 1, y = 0$. For $z = 2k > 2$, we must have

$$5^x 7^y = (3^k - 2)(3^k + 2).$$

Since the factors on the right differ by 4, they can't both be divisible by 5 or both divisible by 7. Moreover, if both 5 and 7 were to divide the same factor, the other factor would be 1, which would lead to

$$3^k - 2 = 1, \text{ so to } k = 1.$$

Hence 5^x must be equal to one of the factors and 7^y the other. But

$$5^x \geq 3^k - 2 \geq 3^2 - 2 = 7,$$

so $x \geq 2$. On the other hand, every power of 5 beyond the first ends with 25. A suitable power of 7 then would have to end with 21 or 29. But the powers of 7 end with 07, 49, 43 or 01. this shows that the solution $(1, 0, 2)$ is unique.

6. Differentiate the binomial

$$\sum_{k=0}^n \binom{n}{k} (-z)^k = (1 - z)^n$$

once and evaluate the result at 1. We get

$$\sum_{k=1}^n k \binom{n}{k} (-1)^{k-1} = \begin{cases} 0, & n \neq 1, \\ 1, & n = 1. \end{cases}$$

7. Since $\sum_{m=-\infty}^{\infty} |a_m| =: \sigma < \infty$, the series $\sum_{m=-\infty}^i n f t y a_m$ converges. Denote its sum by S . Let

$$C_n := \frac{1}{2n+1} \sum_{m=-\infty}^{\infty} |a_{m-n} + a_{m-n+1} + \dots + a_{m+n}|.$$

Let us show that $\lim_{n \rightarrow \infty} C_n = |S|$. Let ε be an arbitrary positive number. There exists $M \in \mathbb{N}$ such that

$$\sum_{|m| > M} |a_m| < \varepsilon.$$

Hence, for $n \geq M$,

$$(2n+1)C_n = \sum_{|m| > n+M} |\dots| + \sum_{n+M \geq |m| > n-M} |\dots| + \sum_{|m| \leq n-M} |\dots|.$$

Now,

$$\sum_{|m|>n+M} |a_{m-n} + \cdots + a_{m+n}| \geq \sum_{|m|>n+M} (|a_{m-n}| + \cdots + |a_{m+n}|) \leq (2n+1) \sum_{|m|>M} |a_m| \leq (2n+1)\varepsilon,$$

$$\sum_{n+M \geq |m| > n-M} |a_{m-n} + \cdots + a_{m+n}| \leq \sum_{n+M \geq |m| > n-M} \sigma \leq 4M\sigma.$$

Since $m - n \leq -M \leq M \leq m + n$ if $|m| \leq n - M$, it follows that

$$\left| |a_{m-n} + \cdots + a_{m+n}| - |S| \right| \leq \sum_{|m|>M} |a_m| < \varepsilon.$$

Combining these inequalities, we get

$$\begin{aligned} & \left| \sum_{|m| \leq n-M} |a_{m-n} + \cdots + a_{m+n}| - (2n - 2M + 1)|S| \right| \\ & \leq \sum_{|m| \leq n-M} \left| |a_{m-n} + \cdots + a_{m+n}| - |S| \right| \leq (2n - 2M + 1)\varepsilon, \end{aligned}$$

$$\left| (2n + 1)C_n - (2n - 2M + 1)|S| \right| \leq (2n + 1)\varepsilon + 4M\sigma + (2n - 2M + 1)\varepsilon.$$

Dividing by $2n + 1$ and taking the limit as $n \rightarrow \infty$, we get

$$\limsup_{n \rightarrow \infty} |C_n - |S|| \leq 2\varepsilon.$$

Since ε was an arbitrary positive number, we conclude that

$$\lim_{n \rightarrow \infty} C_n = |S|.$$

8. **Answer:** If $0 \leq c \leq 1/4$, then f is constant. If $c > 1/4$, then f is uniquely determined as described below by its restriction to the interval $[0, c]$ which may be any continuous function satisfying $f(0) = f(c)$.

Solution. The functional equation implies $f(x) = f(x^2 + c) = f((-x)^2 + c) = f(-x)$. Conversely, if f satisfying the functional equation for $x \geq 0$ and is even, then it satisfies the functional equation for all x . So we can look at $x \geq 0$ only.

Case 1: $0 \leq c \leq 1/4$. The zeros a, b of the polynomial $x^2 - x + c$ are real and satisfy $0 \leq a \leq b$. Let us show that $f(x) = f(a)$ for all x . Suppose $0 \leq x < a$. Define $x_0 := x$ and run the iteration $x_{n+1} := x_n^2 + c$. If $x_n < a$, then $x_{n+1} - x_n = x_n^2 - x_n + c > 0$ and $x_{n+1} = x_n^2 + c < a^2 + c = a$, so $x_n < x_{n+1} < a$ for all $n \in \mathbb{N}$. Hence (x_n) converges to a limit L not exceeding a that satisfies $L^2 + c = L$, which implies $L = a$. But $f(x_n) = f(x_0)$ for all n , we get $f(x) = f(a)$ by continuity of f .

If $a < x < b$, we argue similarly. Defining (x_n) as above, we get $a < x_{n+1} < x_n$, since $x_{n+1} - x_n = x_n^2 - x_n + c < 0$ and $x_{n+1} = x_n^2 + c > a^2 + c = a$. So again $f(x) = f(a)$ for $a < x < b$. By continuity, $f(b) = f(a)$ as well.

Finally, if $x > b$, take $x_0 := x$ and define $x_{n+1} := \sqrt{x_n - c}$. We get $b < x_{n+1} < x_n$, so (x_n) converges to b , and $f(x) = f(b) = f(a)$.

Case 2: $c > 1/4$. Then $x^2 + c > x$ for all x . Let g be any continuous function on $[0, c]$ satisfying $g(0) = g(c)$. Extend g to a continuous function f by setting $f(x) = g(x)$ on $[0, c]$,

$f(x) = g(\sqrt{x-c})$ for $c \leq x < c^2 + c$, $f(x) = g(\sqrt{\sqrt{x-c}-c})$ for $c^2 + c \leq x \leq (c^2 + c)^2 + c$, and so on. This produces a continuous function satisfying the functional equation.

9. Partition the set S into subsets A and B by putting into A all the numbers that do not divide any other number in S . If A is infinite, then (a) holds and we are done.

Suppose then that A is finite. Then B is infinite. Remove into a subset C all elements of B that divide any element of A . Because A is finite, the set of all its possible divisors is also finite. Thus the complementary subset $D := B \setminus C$ must be infinite.

Consider an integer k in D . Not being in A , k must divide another element t of S . This element t cannot belong to A , since in that case its divisor k would have been put into C . Also, if t were to belong to C , then so would be k . Thus each element k of D must divide another element t of D itself. So, beginning with any element k_1 of D , build an infinite chain

$$k_1 \mid k_2 \mid \dots \mid k_n \mid \dots, \quad k_n \in D \text{ for all } n \in \mathbb{N}.$$

10. There exists an orthogonal matrix (l_{ij}) that makes the quadratic form $x'Ax$ into a sum of squares:

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j = \sum_{m=1}^n s_m (l_{m1} x_1 + \dots + l_{mn} x_n)^2$$

where the s_m 's are eigenvalues of A . Therefore,

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij} x_i x_j = \sum_{m=1}^n s_m \left(\sum_{i=1}^n \sum_{j=1}^n b_{ij} (l_{mi} x_i) (l_{mj} x_j) \right) \geq 0$$

because each summand in the outer sum is nonnegative thanks to the nonnegative definiteness of B and nonnegativity of the s_m 's.

11. The function x strictly increases in a neighborhood of $t = 0$, hence it is enough to prove that \dot{x} changes sign exactly once. Consider restrictions of the hyperbolas $\dot{x} = 0$ and $\dot{y} = 0$ to the first quadrant $x, y \geq 0$:

$$\dot{x} = 0 \Leftrightarrow a_3 x \left(y - \frac{a_2}{a_3} \right) + a_1 = 0, \quad \dot{y} = 0 \Leftrightarrow -a_3 \left(x + \frac{1}{a_3} \right) \left(y - \frac{a_4}{a_3} \right) - \frac{a_4}{a_3} = 0.$$

Denote their intersection by (x_0, y_0) . The trajectory of the solution cannot leave the region bounded by the curves $\dot{x} = 0$, $\dot{y} = 0$, $y \geq y_0$. Therefore, it is enough to prove that $y(t) > y_0$ for $t \geq 0$, and the trajectory intersects the curve $\dot{x} = 0$. If the trajectory intersecting the half-line $x = x_0$, $y > y_0$ leaves region D bounded by the lines $x = 0$, $x = x_0$, $l : (a_4 - a_2)x - y + a_1 = 0$, then it must intersect the curve $\dot{x} = 0$, namely in the opposite case

$$y(t) \downarrow z_1, \quad x(t) + y(t) \downarrow \text{const} > 0, \quad \text{and so } x(t) \nearrow x_1 \text{ as } t \rightarrow \infty.$$

This implies that $x_1 = x_0$ and $y_1 = y_0$ because the point (x_1, y_1) must lie on the curves $\dot{x} = 0$, $\dot{y} = 0$. But this is a contradiction for $x(t) > x_0$ for large t .

Next, show that the trajectory must indeed intersect the half-line $x = x_0$, $y > y_0$. Write the function \dot{y}/\dot{x} in the form

$$\frac{\dot{y}}{\dot{x}} = \frac{-a_3xy + a_4x - y}{a_3xy - a_2x + a_1} = -1 + \frac{(a_4 - a_2)x - y + a_1}{a_3xy - a_2x + a_1}.$$

We see that the values of \dot{y}/\dot{x} on the line l would be equal to -1 , which is bigger than $a_4 - a_2$. Hence the trajectory does not intersect the line l . We can also see that the function \dot{y}/\dot{x} is a decreasing function of x in the region $D' := \{(x, y) \in D : y < a_2/a_3\}$ along the lines $(a_4 - a_2)x - y + \text{const} = 0$ ($a_1 < \text{const}$). Since

$$\left. \frac{\dot{y}}{\dot{x}} \right|_{x=x_0} = \frac{-a_3x_0y + a_4x_0 - y}{a_3x_0y - a_2x_0 + a_1} = \frac{-a_3x_0(y - y_0) - (y - y_0)}{a_3x_0(y - y_0)} = - \left(1 + \frac{1}{a_3x_0} \right),$$

it follows that

$$\left. \frac{\dot{y}}{\dot{x}} \right|_{(x,y) \in D'} \geq - \left(1 + \frac{1}{a_3x_0} \right).$$

Now we show that

$$1 + \frac{1}{a_3x_0} < a_2 - a_4.$$

Direct calculations give the formula

$$x_0 = \frac{(a_1a_3 - a_2) + \sqrt{(a_1a_3 - a_2)^2 + 4a_1a_3(a_2 - a_4)}}{2a_3(a_2 - a_4)},$$

from which $a_3x_0 \geq (a_1a_3 - a_2)/(a_2 - a_4)$ follows. So, to deduce the required inequality, it is enough to verify that

$$a_1a_3 - a_2 > \frac{1}{1 - \frac{1}{a_2 - a_4}},$$

which is indeed so by the assumptions of the problem.

Now let l' be the line through the point of intersection of l and the horizontal line $y = a_2/a_3$ with slope between $-(a_2 - a_4)$ and $-(1 + 1/a_3x_0)$. Since

$$\left. \frac{\dot{y}}{\dot{x}} \right|_{(x,y) \in l' \cap D'} \geq - \left(1 + \frac{1}{a_3x_0} \right),$$

the trajectory coming from the region D cannot intersect l' from above. If it does not intersect the line $x = x_0$ either, then $(x(t), y(t)) \rightarrow (x_1, y_1) \neq (x_0, y_0)$ as $t \rightarrow \infty$, a contradiction. This completes the proof.

12. If T is self-adjoint, then it is an orthogonal projector, i.e., $Tx \perp (I - T)x$ for all x . So,

$$\|Tx\|^2 + \|(I - T)x\|^2 = \|x\|^2,$$

hence $\|T\| \leq 1$. If $T \neq 0$, then $Tx = x$ for some $x \neq 0$, hence $\|T\| \geq 1$, hence $\|T\| = 1$.

Conversely, suppose $\|T\| \leq 1$. For any element x of the space and any scalar μ , we have

$$T(\mu Tx - (x - Tx)) = \mu Tx,$$

therefore

$$|\mu|^2 \|Tx\|^2 \leq \|\mu Tx - (x - Tx)\|^2 = -2\operatorname{Re}(\mu \langle Tx, x - Tx \rangle) + |\mu|^2 \|Tx\|^2 + \|x - Tx\|^2.$$

Consequently,

$$\operatorname{Re}(\mu \langle Tx, x - Tx \rangle) \leq \frac{1}{2} \|x - Tx\|^2.$$

But this is possible for all μ if and only if

$$\langle Tx, x - Tx \rangle = 0.$$

This implies that $T = T^*T = T^*$.