Solutions to homework #7.

Solutions to problems 5 and 12 are provided by Boris Bukh.

1. The given surface is a quadric cone containing the three coordinate axes. Indeed, it is star-shaped and its intersections with the planes $x+y+z=c$ are circles:

$$x^2+y^2+z^2=(x+y+z)^2-2(xy+yz+xz)=c^2-0=c^2.$$

These are exactly the planes that cut the cone in a circle. To cut a cone in a parabola, a plane must be parallel but must not include a generator of the cone. In other words, it must be parallel to a plane tangent to the cone.

The plane tangent to our cone at (x_0,y_0,z_0) has the form

$$(z_0+y_0)x+(x_0+z_0)y+(x_0+y_0)z=0.$$

A plane parallel to it is therefore of the form

$$ax+by+cz=d,\quad d\neq 0$$

where $a=y_0+z_0$, $b=x_0+z_0$, $c=x_0+y_0$. Now, (x_0,y_0,z_0) lies on the cone if and only if

$$a^2+b^2+c^2-2(ab+bc+ac)=0.$$

Answer: (1) $x+y+z=\text{const}\neq 0$, (2) $ax+bc+cz=\text{const}\neq 0$ where $a^2+b^2+c^2-2(ab+bc+ac)=0$,

2. Given two integers $n\neq 0$ and p, one of the integers

$$p,\ \ p+1,\ \ldots,\ p+n-1 \tag{1}$$

is a multiple of n. Given n, choose $p:=1234567890\cdot 10^k$ where k is large enough so that $10^k>n$. Then the integers (1) all have decimal representations beginning with 1234567890 and one of them is a multiple of n.

3. With $x_0:=\sqrt{7}$, $x_1:=\sqrt{7-\sqrt{7}}$, we have

$$x_{n+2}=f(x_n):=\sqrt{7-\sqrt{7+x_n}},\qquad n\in\mathbb{Z}_+.$$

Let us show that f is a proper contraction on the interval $[0,7]$. Indeed, the interval $[0,7]$ is seen directly to be invariant under f, and

$$|f'(x)|=\left|\frac{-1}{4\sqrt{7-\sqrt{7+x}}\cdot\sqrt{7+x}}\right|\leq\frac{1}{4\sqrt{7-\sqrt{14}}\cdot\sqrt{7}}=:\alpha<1$$

whenever $x\in[0,7]$. The (unique) fixed point of f on the interval $[0,7]$ is 2, so by the mean value theorem

$$|f(x)-2|=|f(x)-f(2)|=|f'(\xi)|\cdot|x-2|\quad\text{for some }\ \xi\in[0,7],$$

so both subsequences (x_{2n}) and (x_{2n+1}), and therefore the sequence (x_n) as well, converge to the limit 2.

4. According to the rules, any 400 consecutive Gregorian years consist of 303 ordinary years and 97 leap years making a total of $400 \cdot 365 + 97$ days.

$$400 \cdot 365 + 97 \equiv 1 \cdot 1 - 1 \equiv 0 \pmod 7,$$

there is an integral number of weeks in 400 years. Therefore, the day of the week on which Christmas (or any other calendar date) falls repeats in circls of 400. If there are N years when Christmas falls on Wednesday, then $N/400$ is the corresponding probability. But $N/400 \neq 1/7$ for any integer N.

5. The proof is by induction of n. The base case $n = 1$ is straightforward. The induction step is justified by the following derivation:

$$\sum_{j=1}^{n}\left(\sum_{k=j}^{n}\frac{1}{k}\right)^2 - \sum_{j=1}^{n-1}\left(\sum_{k=j}^{n-1}\frac{1}{k}\right)^2 = \frac{1}{n^2} + \sum_{j=1}^{n-1}\left(\left(\sum_{k=j}^{n}\frac{1}{k}\right)^2 - \left(\sum_{k=j}^{n-1}\frac{1}{k}\right)^2\right)$$
$$= \frac{1}{n^2} + \sum_{j=1}^{n-1}\frac{1}{n}\left(2\left(\sum_{k=j}^{n}\frac{1}{k}\right) - \frac{1}{n}\right) = \frac{1}{n^2} + \frac{1}{n}\left(-\frac{n-1}{n} + 2\sum_{k=1}^{n}\sum_{j=1}^{\max(k,n-1)}\frac{1}{k}\right)$$
$$= \frac{1}{n^2} + \frac{1}{n}\left(-\frac{n-1}{n} + 2\sum_{k=1}^{n} k \cdot \frac{1}{k} - 2 \cdot \frac{1}{n}\right) = 2 - \frac{1}{n}.$$

6. Let us show that there are infinitely many integers n for which

$$\frac{a_1 + a_{n+1}}{a_n} > 1 + \frac{1}{n}.$$

This will immediately imply that

$$\limsup_{n\to\infty}\left(\frac{a_1 + a_{n+1}}{a_n}\right)^n \geq \limsup_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e.$$

We will prove the claim by reductio ad absurdum. Suppose the claim fails, so that, for some $k \in \mathbb{N}$ and for all $n \geq k$,

$$\frac{a_1 + a_{n+1}}{a_n} \leq \frac{n+1}{n}.$$

Hence

$$\frac{a_n}{n} \geq \frac{a_1}{n+1} + \frac{a_{n+1}}{n+1},$$

therefore

$$\frac{a_k}{k} \geq \frac{a_1}{k+1} + \frac{a_{k+1}}{k+1} \geq \frac{a_1}{k+1} + \frac{a_1}{k+2} + \frac{a_{k+2}}{k+2} \geq \cdots \geq a_1\left(\sum_{j=k+1}^{p}\frac{1}{j}\right) + \frac{a_p}{p}$$

for any $p \geq k$. But the harmonic series diverges, so the sums $\sum_{j=k+1}^{p} 1/j$ are unbounded, so cannot be smaller than a fixed number a_k/k. This proves our claim.

7. An arithmetic progression with zeroth term $a_0 > 0$ and difference $d \in \mathbb{N}$ is the set

$$\{a_0 + dk : k \in \mathbb{Z}_+\}.$$

Setting $k := 10^m$ and $k := 10^{m+1}$ for m large enough to guarantee $10^m > a_0$ produces two numbers with the same nonzero digits. If both d and a_0 are negative, the same reasoning applies with $10^m > |d|$ replacing $10^m > d$. Finally, if the signs of a_0 and d are opposite, we can consider only the part of the sequence that has the same sign as d.

Remark: This shows, in fact, that any integer arithmetic sequence contains *infinitely many* numbers with a property even stronger than desired.

8. 1) Yes. In step 1, make a 2×2 square by putting together 4 squares with side 1. In step 2, make a 4×4 square by adding 3 more 2×2-squares so that the previously covered square is in the south-west corner of the new configuration. In step 3, make an 8×8 square by adding 3 more 4×4 squares so as the previously covered square is in the north-east corner of the new configuration, and so on. So, in step n add 3 squares of size $2^{n-1} \times 2^{n-1}$ so as to cover a square of size $2^n \times 2^n$ keeping the previously covered area in the SW or NE corner, depending on the parity of n. This uses 4 squares of size 1×1 and 3 squares of each other available size.

2) No. First note that $1 + 2 + \cdots + 2^n < 2^{n+1}$, so at least one side of each square should be on its boundary with a larger square. Consider an arbitrary square $ABCD$. Its side AB must be next to a larger square, hence at least one of the vertices A and B (say, B) must lie on a side of a larger square. But then C must lie on a side of a square also larger than $ABCD$ that is next to the side BC, and so on. So, each vertex of each square must lie on a side of a larger square.

Now consider the square of minimal size used in a hypothetical covering of the whole plane. It is surrounded by exactly four larger squares. Consider its vertex that is on the side of the largest of the four squares. The property we just derived is then violated with respect to that larger square. Contradiction.

9. Notice that

$$(I - P - Q)Q = Q - PQ - Q^2 = -PQ = P - P^2 - PQ = P(I - P - Q).$$

Since $I - P - Q$ is invertible, the matrices P and Q are in fact similar, hence in particular have the same rank.

10. Suppose that $|f''(x)| \leq c|f'(x)|$ for $x \geq x_0$. We first show that

$$|f'(x+t)| \leq \frac{|f'(x)|}{1 - ct} \quad \text{for } x \geq x_0, \ \ t < 1/c. \tag{2}$$

We may assume that $|f'(x+t)| > |f'(x)|$, since otherwise (2) holds trivially. Let

$$t_0 := \min\{t' : t' > 0, |f'(x+t')| = |f'(x+t)|\}.$$

Since the function $|f'(\xi)|$ is continuous and does not intersect the horizontal line of height $|f'(x+t)|$ for $\xi \in [x, x+t_0)$ while it remains under that line at the point x, we see that $|f'(\xi)| \leq |f'(x+t)|$ on the whole interval $[x, x+t_0]$. By the mean value theorem,

$$|f'(x+t)| - |f'(x)| \leq |f'(x+t_0)| - |f'(x)| = t_0|f''(\xi)| \leq t_0 c|f'(\xi)| \leq t_0 c|f'(x+t)|,$$

which gives (2). Now (2) implies that $f'(x') = 0$ once $f(x) = 0$ and $x' \geq x \geq x_0$. But this is impossible, since in that case $e^{-x}f(x)$ would tend to 0. So, f' is never zero for $x \geq x_0$, and since $e^{-x}f(x) \to 1$, the function f cannot be monotone decreasing, so

$$f'(x) > 0 \quad \text{for all} \quad x \geq x_0. \tag{3}$$

From (2) and (3) we obtain

$$(1-ct)f'(x+u) \leq f'(x) \leq \frac{1}{1-ct}f'(x-u) \quad \text{whenever} \;\; x \geq x_0 + 1/c, \;\; t < 1/c, \;\; 0 \leq u \leq t.$$

Hence, by integration,

$$(1-ct)\left(f(x+t) - f(x)\right) \leq tf'(x) \leq \frac{1}{1-ct}\left(f(x) - f(x-t)\right),$$

that is,

$$\frac{1-ct}{t}\left(\frac{f(x+t)}{e^{x+t}}e^t - \frac{f(x)}{e^x}\right) \leq \frac{f'(x)}{e^x} \leq \frac{1}{t(1-ct)}\left(\frac{f(x)}{e^x} - \frac{f(x-t)}{e^{x-t}}e^{-t}\right). \tag{4}$$

Now, for fixed t,

$$\lim_{x\to\infty}\left(\frac{f(x+t)}{e^{x+t}}e^t - \frac{f(x)}{e^x}\right) = e^t - 1.$$

Hence

$$\liminf_{x\to\infty}\frac{f'(x)}{e^x} \geq (1-ct)\frac{e^t-1}{t},$$

so

$$\liminf_{x\to\infty}\frac{f'(x)}{e^x} \geq \lim_{t\to 0+}(1-ct)\frac{e^t-1}{t} = 1.$$

Finally, it follows similarly from the right inequality in (4) that

$$\limsup_{x\to\infty}\frac{f'(x)}{e^x} \leq \lim_{t\to 0+}\left(\frac{1}{1-ct}\cdot\frac{1-e^{-t}}{t}\right) = 1.$$

This finishes the proof.

11. The proof is by induction. The case $n = 2$ is straightforward. For $n > 2$, suppose the proposition is true for any system of points in a square of side length a containing $k < n$ points, i.e., that

$$\sum_{j=1}^{k} r_j^2 \leq 4a^2.$$

Divide the square of side length 1 into 4 equal squares by the medians of its sides. Denote the small squares by N_j, $j = 1, 4$, say, counting counterclockwise. If a point lies on the boundary of the smaller squares, assign it to any square it belongs to in an arbitrary way.

Case 1. None of the smaller squares contains exactly one point. If all points are in one subsquare, then the points $x_1, \ldots, x_{n-1}$ satisfy

$$\sum_{j=1}^{n-1} r_j^2 \leq 4 \left(\frac{1}{2}\right)^2 = 1.$$

Since $r_n^2 \leq 2$, we get $\sum_{j=1}^{n} r_j^2 \leq 3$. If each small square contains more than 1 point, then

$$\sum_{k=1}^{4} \sum_{x_j \in N_k} r_j^2 \leq 4 \left(\frac{1}{2}\right)^2 = 1.$$

Case 2. There is exactly one smaller square (say, N_1) that contains 1 point x_1. If $r_1 \leq 1$, then the previous argument is enough. If $r_1 > 1$, then the generous upper bound $r_1^2 \leq 2$ is also enough if one of the small squares is empty. So, assume that each square contains a point. From this it follows right away that $r_1^2 \leq 5/4$. Divide N_4 into four even smaller squares similarly as we did with the big square and so that N_{41} and N_{42} are next to N_1. As $r_1 > 1$, the squares N_{41} and N_{42} cannot contain any points. If the number of points contained in N_{4j}, $j = 3, 4$ is not one, then by the induction hypothesis

$$\sum_{x_j \in N_{4j}} r_j^2 \leq \frac{5}{16}, \qquad j = 3, 4. \tag{5}$$

If N_{4j} contains exactly one point, then its distance to any other point in N_4 is at most $\sqrt{5}/4$, so (5) holds as well. So, in any case,

$$\sum_{j=1}^{n} r_j^2 = r_1^2 + \sum_{x_j \in N_2} r_j^2 + \sum_{x_j \in N_3} r_j^2 + \sum_{x_j \in N_4} r_j^2 \leq \frac{5}{4} + 1 + 1 + 2 \cdot \frac{5}{16} < 4.$$

Case 3. The number of points is equal to 1 in exaclty two smaller squares. In this case, the squares can be coupled in such a way that squares in a couple of neighbors and exactl one square in a couple contains exactly one point. Suppose, for example, that N_1 and N_4 form such a couple. Then it can be argued as in Case 2 that

$$\sum_{x_j \in N_1 \cup N_4} r_j^2 \leq 2,$$

which implies the desired inequality.

Case 4. Exaclty 3 squares, say N_1, N_2, N_3, contain exactly 1 point. Subdivide N_4 into four smaller squares as in Case 2. If the rectanges $N_{41} \cup N_{42}$ and $N_{43} \cup N_{42}$ (with N_{43} next to N_3), then

$$r_1^2,\ r_3^2 \leq \left(\frac{3}{4}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{13}{16}.$$

Consequently,

$$\sum_{j=1}^{n} r_j^2 = r_1^2 + r_2^2 + r_3^2 + \sum_{x_j \in N_4} r_j^2 \leq \frac{13}{16} + \frac{5}{4} + \frac{13}{16} + 1 < 4.$$

If $N_{41} \cup N_{42}$ contains no point but N_{43} and N_{44} do contain a point, then $r_3^2 \leq 13/16$, $r_1^2 \leq 1 + 1/16$, and (5) is valid. Hence

$$\sum_{j=1}^{n} r_j^2 = r_1^2 + r_2^2 + r_3^2 + \sum_{x_j \in N_4} r_j^2 \leq \frac{17}{16} + \frac{5}{4} + \frac{13}{16} + 2 \cdot \frac{5}{16} < 4.$$

If neither N_{41} nor N_{42} contains a point and one of N_{43} and N_{44} is also empty, then by the induction hypothesis on the other (nonempty) square, we get

$$\sum_{j=1}^{n} r_j^2 = r_1^2 + r_2^2 + r_3^2 + \sum_{x_j \in N_4} r_j^2 \leq \frac{5}{4} + \frac{5}{4} + \frac{5}{4} + \frac{1}{4} = 4.$$

Case 5. Each of the small squares contains one point. Let $x_j \in N_j$, $j = 1, 2, 3, 4$. We prove the stronger inequality

$$\operatorname{dist}(x_1, x_2)^2 + \operatorname{dist}(x_2, x_3)^2 + \operatorname{dist}(x_3, x_4)^2 + \operatorname{dist}(x_4, x_1)^2 \leq 4.$$

the left-hand side of this inequality is a convex function, so may attain its maximum only at the extremal points on the boundary of its domain. An extremal point corresponds to a configuration where each x_j is a vertex of the corresponding square N_j. In these cases the desired inequality can be verified directly, so we are done.

12. The indices of elements of a submatrix are elements of a Cartesian product $A \times B$ where $A, B \subset [n]$. The family $\mathcal{A}$ of all A's as $A \times B$ runs over all submatrices in the family is a k-uniform intersecting family in $2^{[n]}$. By the Erdős-Ko-Rado theorem, the cardinality of $\mathcal{A}$ does not exceed $\binom{n-1}{k-1}$. Similar reasoning applies to the family of all B's. Therefore the cardinality of the family of submatrices does not exceed $\binom{n-1}{k-1}^2$.