Solutions to homework #8.

Solutions to problems 2, 3, 4, 6 are provided by Boris Bukh.

1. Denote the center of the square by O, its projections on the X- and Y-axes by A and B and the vertices of the square lying on the X- and Y-axes by A' and B', respectively. Since the diagonals of a square are equal and meet at a right angle, the right triangles OAA' and OBB' are equal. Hence their legs OA and OB are equal. In other words, the x- and y-coordinates of the center are always equal. The length of OA cannot exceed that of OA', which is $\sqrt{2}a$, and is at least a. Both values are attained, the first when the square is at an angle $\pi/4$ to the axes, the second when it sits in the corner, and by continuity all the intermediate values are attained as well.

Answer: The segment $\{(t,t) : t \in [a, \sqrt{2}a]\}$.

2. Let us first show that the condition $g(x_0) = 0$ implies $g(x_0 + \tau) = 0$ for all $|\tau| \leq 1/3$. Due to the initial condition $g(0) = 0$ this implies that g is identically zero. Let $M := \sup_{\tau\in[-1/3,1/3]} |g(x_0 + \tau)|$. Since $[-1/3, 1/3]$ is compact, there exists $\tau_* \in [-1/3, 1/3]$ such that $|g(x_0 + \tau_*)| = M$. We have

$$\begin{aligned} M &= |g(x_0+\tau_*)| = \left|g(x_0) + \int_0^{\tau_m} g'(x_0+\tau)\,d\tau\right| = \left|\int_0^{\tau_*} g'(x_0+\tau)\,d\tau\right| \leq \int_0^{\tau_*} |g'(x_0+\tau)|\,d\tau \\ &\leq \int_{-1/3}^{1/3} |g'(x_0+\tau)|\,d\tau \leq \int_{-1/3}^{1/3} |g(x_0+\tau)|\,d\tau \leq \frac{2}{3}M \end{aligned}$$

which implies that $M = 0$.

3. Since $b_n \geq 0$, the following change in the order of summation is justified

$$\sum_{n=1}^{\infty} nb_n = \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} b_n = \sum_{m=1}^{\infty}\sum_{n=m}^{\infty}(a_n - 2a_{n+1} + a_{n+2}) = \sum_{m=1}^{\infty}(a_m - a_{m+1}) = a_1.$$

4. **Answer:** The series diverges. **Proof.** If $\exp(\exp(2\pi k - \pi/3)) \leq n \leq \exp(\exp(2\pi k + \pi/3))$, then $\cos(\ln\ln n) \geq 1/2$. Hence

$$\begin{aligned} \sum_{n=\lceil\exp(\exp(2\pi k-\pi/3))\rceil}^{\lfloor\exp(\exp(2\pi k+\pi/3))\rfloor} \frac{\cos(\ln\ln n)}{\ln n} &\geq \frac{1}{2} \sum_{n=\lceil\exp(\exp(2\pi k-\pi/3))\rceil}^{\lfloor\exp(\exp(2\pi k+\pi/3))\rfloor} \frac{1}{\ln n} \\ &\geq \frac{1}{2} \sum_{n=\lceil\exp(\exp(2\pi k-\pi/3))\rceil}^{\lfloor\exp(\exp(2\pi k+\pi/3))\rfloor} \frac{1}{\exp(2\pi k - \pi/3)} \\ &= \frac{1}{2} \frac{\lfloor\exp(\exp(2\pi k+\pi/3))\rfloor - \lceil\exp(\exp(2\pi k-\pi/3))\rceil + 1}{\exp(2\pi k - \pi/3)} \end{aligned}$$

which diverges as $k \to \infty$. Therefore, the sequence of partial sums of the series fails to satisfy the Cauchy criterium of convergence, i.e., the series diverges.

5. 1) Note that

$$a_{n+1}-1=a_n(a_n-1)=a_na_{n-1}(a_{n-1}-1)=\cdots=a_na_{n-1}\cdots a_1.$$

This implies that $(a_n,a_m)=1$ for $m\neq n$.

2) Now show that

$$\sum_{k=1}^{n-1}\frac{1}{a_k}=1-\frac{1}{a_n-1}\quad\text{for}\quad n>1$$

by induction on n. The case $n=2$ can be checked by direct calculation. To justify the induction step from n to $n+1$, just notice that, by the observation made in 1),

$$\frac{1}{a_n-1}-\frac{1}{a_{n+1}-1}=\frac{1}{\prod_{j<n}a_j}-\frac{1}{\prod_{j<n+1}a_j}=\frac{a_n-1}{\prod_{j<n+1}a_j}=\frac{\prod_{j<n}a_j}{\prod_{j<n+1}a_j}=\frac{1}{a_n}.$$

Since $a_{n+1}-1=a_n(a_n-1)$ and $a_1=2$, we see that the sequence (a_n) monotonically increases to infinity. Hence $\sum_{k=1}^{n-1}1/a_k=1-1/(a_n-1)$ converges to 1 as $n\to\infty$.

6. $\sin 17x=\cos(\pi/2-17x+8\pi)=\cos(17(\pi/2-x))=f(\cos(\pi/2-x))=f(\sin x)$.

7. For simplicity, let us wrap around the indices, so that $x_0:=x_n$, $x_{-1}:=x_{n-1}$, …, and $x_{n+1}:=x_1$, $x_{n+2}:=x_2$, …. Now write each fraction as

$$\frac{x_j}{x_{j+1}+x_{j+2}}=\frac{x_j+\frac12x_{j+1}}{x_{j+1}+x_{j+2}}+\frac{\frac12x_{j+1}+x_{j+2}}{x_{j+1}+x_{j+2}}-1.$$

We need to show that the sum of all the new fractions is bigger than $\sqrt2n$. Let us combine each fraction that appears second with the fraction listed first in the next sum. We get the sums

$$\frac{\frac12x_j+x_{j+1}}{x_j+x_{j+1}}+\frac{x_j+\frac12x_{j+1}}{x_{j+1}+x_{j+2}}.$$

By the AM-GM inequality, each such sum is bounded below by

$$2\sqrt{\frac{\frac12x_j+x_{j+1}}{x_j+x_{j+1}}\cdot\frac{x_j+\frac12x_{j+1}}{x_{j+1}+x_{j+2}}}=2\sqrt{\frac{(x_j+x_{j+1})^2+\frac12x_jx_{j+1}}{2(x_j+x_{j+1})(x_{j+1}+x_{j+2})}}>\sqrt2\sqrt{\frac{x_j+x_{j+1}}{x_{j+1}+x_{j+2}}}.$$

Finally, apply the AM-GM inequality again to the obtained lower bounds. The resulting product of n fractions is 1, so the final lower bound is $\sqrt2n$, hence the original sum is bounded by $\sqrt2n-n$, as we were asked to prove.

8. Let z_j, $j=1,\dots,n$ denote the points on the circle. After rotation if necessary, we may assume that $(-1)^nz_1\cdots z_n=1$. Consider the polynomial

$$(z-z_1)\cdots(z-z_n)=:z^n+q(z)+1.$$

We want to show that $q(z)=0$. To this end, first observe that q contains only the first through the $(n-1)$st powers of z, so

$$\sum_{j=1}^{n}q(\omega^j)=0\quad\text{where}\quad\omega:=e^{2\pi\mathrm{i}/n}.$$

If $q(\omega^j) = 0$ for all $j = 1, \ldots, n$, then q is identically zero, since its degree does not exceed $n-1$. Otherwise $q(\omega^j) \neq 0$ *and* $\mathrm{Re}[q(\omega^j)] \geq 0$ for some j. But then $|\omega^{jn} + q(\omega^j) + 1| = |2 + q(\omega^j)| > 2$ contrary to the assumption of the problem. Therefore q is identically zero, which finishes the proof.

9. Consider the possible preceeding totals. These are 7 through 12. If the preceeding total was 12, then the current total can be any of the numbers 13 through 18, each of these 6 values with equal probability. If the preceeding total was 11, the current total can be anything from 13 to 17, each of these 5 values with equal probability, and so forth. Since the value 13 is the only one that appears for all preceeding totals, 13 is the most likely running total.

10. The proof is by induction. The cases $n = 1$ and $n = 2$ can be verified directly. Let us denote the trigonometric sum under consideration by $S_n(\alpha)$. Suppose S_{n-1} satisfies the inductive hypothesis but S_n has a minimum $\alpha_0 \in (0, \pi)$ where $S_n(\alpha_0) \leq 0$. Since

$$S_n'(\alpha_0) = \cos\alpha_0 + \cdots + \cos n\alpha_0 = \frac{\sin\left((n+1/2)\alpha_0\right) - \sin\left(\alpha_0/2\right)}{2\sin\left(\alpha_0/2\right)} = 0,$$

we must have $\sin((n+1/2)\alpha_0) = \sin(\alpha_0/2)$, which implies $|\cos((n+1/2)\alpha_0)| = \cos(\alpha_0/2)$. But then

$$S_n(\alpha_0) - S_{n-1}(\alpha_0) = \frac{1}{n}\sin n\alpha_0 = \frac{1}{n}\sin\left(\left(n+\frac{1}{2}\right)\alpha_0\right)\cos\frac{\alpha_0}{2} - \cos\left(\left(n+\frac{1}{2}\right)\alpha_0\right)\sin\frac{\alpha_0}{2} \geq 0,$$

i.e., $S_{n-1}(\alpha_0) \leq S_n(\alpha_0) \leq 0$, which contradicts the inductive hypothesis. Hence $S_n(\alpha) > 0$ for all $\alpha \in (0, \pi)$.

11. Let ϕ_x denote the number of sets A_j containing the point x. Then, for each j,

$$\sum_{x\in A_j} \phi_x = \sum_{i=1}^{n+1} |A_i \cap A_j| = n + \sum_{i\neq j} |A_i \cap A_j|, \tag{1}$$

hence $\sum_{x\in A_j} \phi_x \leq n + nk = n(k+1)$. Summing up over all j, in the left-hand side we get

$$\sum_{j=1}^{n+1} \sum_{x\in A_j} \phi_x = \sum_{x\in A} \sum_{A_j \ni x} \phi_x = \sum_{x\in A} \phi_x^2.$$

Estimating the last sum by Cauchy's inequality, we get

$$\sum_{x\in A} \phi_x^2 \geq |A| \left(\sum_{x\in A} \phi_x/|A|\right)^2 = \frac{1}{|A|}\left(\sum_{i=1}^{n+1} |A_j|\right)^2 = \frac{n^2(n+1)^2}{|A|}.$$

Summing up the right-hand sides of (1) over all j, we obtain $n(n+1)(k+1)$. So,

$$\frac{n^2(n+1)^2}{|A|} \leq n(n+1)(k+1),$$

so

$$|A| \geq \frac{n(n+1)}{k+1}.$$

By the assumption of the problem, the opposite inequality also holds, hence we must have equality (of course, $(k+1)$ must then divide $n(n+1)$).

Answer: $n(n+1)/(k+1)$.

12. First of all, f never vanishes due to the assumption of the problem. Next, f is easily seen to be 3π-periodic. Indeed, since f is even, we have $f''(-x) = f''(x)$, and so

$$\frac{1}{f(-x+3\pi/2)} = f''(-x) + f(-x) = f''(x) + f(x) = \frac{1}{f(x+3\pi/2)},$$

hence $f(x+3\pi/2) = f(-x+3\pi/2)$, hence $f(x) = f(-x+3\pi) = f(x-3\pi)$, the last equality by evenness again. Thus, f is 3π-periodic. Since it is both 2π- and 3π-periodic, it must be π-periodic. So, the assumption of the problem can be rewritten as

$$f''(x) + f(x) = \frac{1}{f(x+\pi/2)}.$$

Now let $g(x) := f(x+\pi/2)$. Note that g is also even:

$$g(-x) = f(-x+\pi/2) = f(x-\pi/2) = f(x+\pi/2) = g(x).$$

In view of the facts $g'(x) = f'(x+\pi/2)$ and $g''(x) = f''(x+\pi/2)$, we can also write

$$f''(x) + f(x) = \frac{1}{g(x)}, \qquad g''(x) + g(x) = \frac{1}{f(x)}.$$

Multiplying the first equality by g, the second by f and subtracting, we get

$$f''g - g''f = (f'g - g'f)' = 0,$$

so the function $f'g - g'f$ is a constant. But the derivative of an even function is odd, so c must be zero. Since g never vanishes, we can now conclude that $(f/g)' = c/g^2 = 0$, hence f/g is constant.

Finally, f is continuous and periodic, hence it takes on its maximum and minimum at some points $x_{\max}$ and $x_{\min}$. So, $g(x_{\min}) = f(x_{\min} + \pi/2) \geq f(x_{\min})$, $g(x_{\max}) = f(x_{\max} + \pi/2) \leq f(x_{\max})$. These inequalities imply that $f/g = 1$, which finishes the proof.