

## Solutions to homework #9.

1. Evaluating the two polynomials at 0 and at 1, we observe that  $a$  must divide 92 and 90, hence must also divide their difference 2. So,  $a = \pm 1$  or  $\pm 2$ . Next observe that  $x^{13} + x + 90$  is positive for positive values of  $x$ , while the first polynomial has a positive root whenever  $a < 0$ . This leaves two possibilities:  $a = 1$  or  $a = 2$ . Now, evaluation at  $-1$  yields the condition  $(a + 2) \mid 88$ , which rules out  $a = 1$ . The polynomial  $x^2 - x + 2$  indeed divides  $x^{13} + x + 90$ , which can be verified by reducing powers of  $x$  in the second polynomial using the relation  $x^2 = x - 2$  from the first polynomial and obtaining zero or by brute force (long division).

2. Consider the convex hull of the five points. If it is a pentagon or a quadrilateral, the result follows immediately. Since no three points are on one line, the only remaining case is when the convex hull is a triangle with vertices among the given points, say,  $A$ ,  $B$  and  $C$ . The other two points  $D$  and  $E$  are in the interior of  $ABC$  and the line  $DE$  meets the triangle twice, not at a vertex. By renaming if necessary, we can assume that  $DE$  meets the sides  $AB$  and  $AC$  but not  $BC$ . Then  $B$ ,  $C$ ,  $D$  and  $E$  are vertices of a convex quadrilateral.

3. By the continuity of the logarithmic function, the question amounts to asking whether the numbers of the form

$$m \ln 2 + n \ln 3, \quad m, n \in \mathbb{Z},$$

are dense in  $\mathbb{R}$ . We use the following theorem.

**Theorem 1.** If  $\alpha, \beta \in \mathbb{R}$ , then the numbers of the form  $m\alpha + n\beta$ ,  $m, n \in \mathbb{Z}$ , are dense in  $\mathbb{R}$  unless  $p\alpha + q\beta = 0$  for some integers  $p$  and  $q$ , not both equal to zero.

Note that  $\ln 2$  and  $\ln 3$  satisfy the assumption of the theorem, since  $2^p 3^q \neq 1$  if  $p$  and  $q$  are integers and not both zero. Indeed, we can assume that  $p$  is nonnegative. If  $q$  is nonnegative, then both have to be equal to zero, since otherwise  $2^p 3^q$  is too large, and if  $q$  is nonpositive, then the same conclusion follows from the uniqueness of factorization into primes. To prove Theorem 1, we start with another theorem.

**Theorem 2.** If  $T$  is a subgroup of the additive group of  $\mathbb{R}$ , then  $T$  is of one of the following types:

- $T = \{0\}$ ,
- $T = h\mathbb{Z}$  for some  $h > 0$ ,
- $T$  is dense in  $\mathbb{R}$ .

**Proof.** Suppose  $T$  contains a least positive number  $h$ . We will prove that  $T$  consists of all multiples of  $h$ . Indeed,  $h\mathbb{Z} \subset T$ . To prove the other inclusion, suppose  $y \in T$ . For some integer  $n$ ,  $n \leq y/h < n + 1$ . Then  $z := y - nh \in T$  and  $0 \leq z < h$ . Since  $h$  is the least positive element of  $T$ , we must have  $z = 0$ , thus,  $y = nh$ . So,  $T = h\mathbb{Z}$ .

Suppose  $T$  contains a positive number  $x$ , but no least positive number. Let  $I := (a, a + \delta)$  be an open interval in  $\mathbb{R}$ . Since  $T$  has infinitely many elements in  $(0, x)$ , two of them, say

$x_1$  and  $x_2$ , are within  $\delta$  of each other. We may assume  $x_1 < x_2$ . Then  $s := x_2 - x_1 \in T$ ,  $0 < s < \delta$ , and some multiple of  $s$  lies in the interval  $I$ . Since  $I$  was arbitrary, this shows that  $T$  is dense in  $\mathbb{R}$ .

To prove now Theorem 1, just observe that the numbers of the form  $m\alpha + n\beta$  form a subgroup  $T$ , with  $\alpha, \beta \in T$ . If  $T$  is not dense, then  $T$  consists of all multiples of some nonnegative number. Then  $\alpha = qh$ ,  $\beta = -qh$ , and  $p\alpha + q\beta = 0$ .

**Answer:** The set  $\{2^m 3^n : m, n \in \mathbb{Z}\}$  is dense in  $P$ .

4. Consider the square  $0 \leq x \leq \beta$ ,  $0 \leq y \leq \beta$ . Choosing a random pair  $(x, y)$  amounts to choosing a random point from the square. We need to determine what fraction of the total area of the square (which is  $\beta^2$ ) is determined by the condition  $|x - y| \geq \alpha$ . The area of interest consists of two triangles, one with vertices  $(\alpha, 0)$ ,  $(\beta, 0)$ ,  $(\beta, \beta - \alpha)$ , corresponding to the inequality  $x \geq y + \alpha$ , and the other with vertices  $(0, \alpha)$ ,  $(0, \beta)$ ,  $(\beta - \alpha, \beta)$ , corresponding to the inequality  $y \geq x + \alpha$ . Each triangle has area  $(\beta - \alpha)^2/2$ , so the combined area of the triangles is  $(\beta - \alpha)^2$ , so the resulting probability is  $(\beta - \alpha)^2/\beta^2$ .

**Answer:**  $(\beta - \alpha)^2/\beta^2$ .

5. Let us denote by  $a_n$  the number of all integers with  $n$  digits that satisfy the assumption of the problem, and by  $a(n, i)$  the number of only those that end with digit  $i$ . Since the adjacent digits differ by 1, we get

$$\begin{aligned} a(n, 1) &= a(n - 1, 2) \\ a(n, 2) &= a(n - 1, 1) + a(n - 1, 3) \\ a(n, 3) &= a(n - 1, 2) + a(n - 1, 4) \\ a(n, 4) &= a(n - 1, 3) + a(n - 1, 5) \\ a(n, 5) &= a(n - 1, 4) \end{aligned}$$

By symmetry,  $a(n, 1) = a(n, 5)$  and  $a(n, 2) = a(n, 4)$ . Now it can be proven by induction that

$$a(2k+1, 1) = a(2k+1, 5) = 2 \cdot 3^{k-1}, \quad a(2k+1, 2) = a(2k+1, 4) = 3^k, \quad a(2k+1, 3) = 4 \cdot 3^{k-1},$$

and

$$a(2k+2, 1) = a(2k+2, 5) = 3^k, \quad a(2k+2, 2) = a(2k+2, 4) = a(2k+2, 3) = 2 \cdot 3^k,$$

which yields

$$a_{2k+1} = 8 \cdot 3^{k-1} + 2 \cdot 3^k, \quad a_{2k+2} = 8 \cdot 3^k,$$

which can be expressed in a single formula as

$$a_n = 8 \cdot 3^{\lfloor (n-2)/2 \rfloor} + (1 - (-1)^n) \cdot 3^{\lfloor n/2 \rfloor}.$$

**Answer:**  $8 \cdot 3^{\lfloor (n-2)/2 \rfloor} + (1 - (-1)^n) \cdot 3^{\lfloor n/2 \rfloor}$ .

6. The function  $S(x) := \sum_{k=1}^{70} \frac{k}{x-k}$  has vertical asymptotes at  $k = 1, \dots, 70$  and is continuous elsewhere. The function jumps from  $-\infty$  to  $\infty$  as it passes through each value  $k$  corresponding to one of its vertical asymptotes. Thus the graph of  $S$  crosses the line  $y = 5/4$  exactly

once on each interval  $(k, k + 1)$  for  $k = 1, \dots, 69$ . In addition,  $\lim_{x \rightarrow \infty} S(x) = 0$ , so there exists one more intersection for  $x > 70$ . Since  $S(x) < 0$  for  $x < 1$ , there is no intersection to the left of the line  $x = 1$ . So, the set of values of  $x$  for which  $S(x) \geq 5/4$  consists of 70 half-open intervals that begin at the points  $x = 1, 2, \dots, 70$ . The length of the  $k$ th interval is  $x_k - k$  where  $x_k$  is the  $k$ th solution to the equation  $S(x) = 5/4$ . The sum of all 70 intervals is therefore

$$\sum_{k=1}^{70} x_k - \sum_{k=1}^{70} k.$$

Now, clearing of fractions in  $S(x) - 5/4 = 0$  yields a polynomial equation of degree 70 of the form

$$-5x^{70} + (4(1 + 2 + \dots + 70) + 5(1 + 2 + \dots + 70))x^{69} + \dots = 0.$$

Therefore the sum of its roots is

$$\frac{9(1 + 2 + \dots + 70)}{5} = (1 + 2 + \dots + 70) + 1988,$$

hence the sum of the intervals is indeed 1988.

7. Let us prove

$$\sum_{n=1}^k \frac{n}{a_1 + a_2 + \dots + a_n} \leq 4 \sum_{n=1}^k \frac{1}{a_n}$$

for all even  $k =: 2t$ . From this inequality the desired conclusion for infinite sums, with  $k = 4$ , will follow immediately. Let  $b_1, \dots, b_k$  be the terms  $a_1, \dots, a_k$  enumerated in increasing order. For  $1 \leq p \leq t$  we have

$$a_1 + a_2 + \dots + a_{2p} \geq a_1 + a_2 + \dots + a_{2p-1} \geq b_1 + b_2 + \dots + b_{2p-1} \geq pb_p,$$

since all terms are positive and the last  $p$  terms are greater than or equal to  $b_p$ . Therefore,

$$\frac{2p-1}{a_1 + \dots + a_{2p-1}} \leq \frac{2p-1}{pb_p} < \frac{2}{b_p}.$$

Also,

$$\frac{2p}{a_1 + \dots + a_{2p}} \leq \frac{2}{b_p}.$$

Hence

$$\frac{2p-1}{a_1 + a_2 + \dots + a_{2p-1}} + \frac{2p}{a_1 + a_2 + \dots + a_{2p}} < \frac{4}{b_p}.$$

Thus

$$\sum_{p=1}^t \left( \frac{2p-1}{a_1 + a_2 + \dots + a_{2p-1}} + \frac{2p}{a_1 + a_2 + \dots + a_{2p}} \right) < \sum_{p=1}^t \frac{4}{b_p}.$$

Rewriting the left-hand sum, we get

$$\sum_{n=1}^k \frac{n}{a_1 + a_2 + \dots + a_n} \leq \sum_{p=1}^t \frac{4}{b_p} < \sum_{p=1}^k \frac{4}{b_p} = 4 \sum_{n=1}^k \frac{1}{a_n}.$$

If the series  $\sum_{n=1}^{\infty}(1/a_n)$  diverges, then the required inequality is satisfied trivially. Otherwise we get

$$\sum_{n=1}^k \frac{n}{a_1 + \cdots + a_n} < 4 \sum_{n=1}^{\infty} \frac{1}{a_n},$$

and hence

$$\sum_{n=1}^{\infty} \frac{n}{a_1 + \cdots + a_n} \leq 4 \sum_{n=1}^{\infty} \frac{1}{a_n}.$$

8. Taking logarithms, we see that the equality is equivalent to

$$\lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{k}{n} \ln k - \frac{n+1}{2} \ln n + \frac{n}{4} \right) = 0$$

or, equivalently,

$$\lim_{n \rightarrow \infty} n \left( -\frac{1}{4} - \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \ln \frac{k}{n} \right) = 0.$$

Consider the function  $f(x) := x \ln x$  on the interval  $[0, 1]$  and notice that  $\int_0^1 f(x) dx = -1/4$  and the last sum is a Riemann sum for  $f$  on that interval. So, we need to check that

$$\Delta_n := \int_0^1 f(x) dx - \frac{1}{n} \sum_{k=1}^n f(k/n) = o(n).$$

Now, by integration by parts,

$$\Delta_n = \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left( \frac{k}{n} - x \right) f(x) dx.$$

For each interval  $[\frac{k-1}{n}, \frac{k}{n}]$ , with  $k > 1$ , we can use the estimate

$$\frac{1}{2n^2} m_k \leq \int_{(k-1)/n}^{k/n} \left( \frac{k}{n} - x \right) f(x) dx \leq \frac{1}{2n^2} M_k$$

where  $m_k$  and  $M_k$  denote the infimum and the supremum of  $f'$  on the interval  $[\frac{k-1}{n}, \frac{k}{n}]$ . For the first interval, one can check directly that  $\lim_{n \rightarrow \infty} n \int_0^1 (\frac{1}{n} - x) f'(x) dx = 0$ . Since  $f'$  is monotone and integrable on  $[0, 1]$ , we thus obtain Riemann sums for  $f'$ , which therefore tend to the same limit:

$$\lim_{n \rightarrow \infty} n \Delta_n = \frac{1}{2} [f(1) - f(0)] = 0.$$

This finishes the proof.

9. Let  $S_n$  be the sum of the cubes of the lengths of the segments formed by the points  $P_1, \dots, P_n$ . Define  $S_0 := 1$ . The sum  $S_n$  is obtained from  $S_{n-1}$  by replacing the term  $(a_n + b_n)^3$  by  $a_n^3 + b_n^3$  and leaving all other terms fixed. Hence

$$S_{n-1} - S_n = (a_n + b_n)^3 - a_n^3 - b_n^3 = 3a_n b_n (a_n + b_n).$$

Therefore

$$1 - S_k = S_0 - S_k = \sum_{n=1}^k 3a_n b_n (a_n + b_n).$$

If we prove that  $\lim_{k \rightarrow \infty} S_k = 0$ , the desired result follows immediately.

To show that  $\lim_{k \rightarrow \infty} S_k = 0$ , let  $t$  be a positive integer. Since the set  $\{P_j\}$  is dense, we can choose an integer  $q$  so large that the set  $\{P_1, \dots, P_q\}$  intersects each of the intervals  $[j/t, (j+1)/t]$ ,  $j = 0, \dots, t-1$ . Suppose  $k \geq q$  and let  $I_0, \dots, I_k$  be the lengths of the intervals determined by the points  $P_1, \dots, P_k$ . Then each of these intervals has length at most  $2/t$ , hence

$$S_k = \sum_{j=0}^k I_j^3 \leq \left(\frac{2}{t}\right)^2 \sum_{j=0}^k I_j = \frac{4}{t^2}.$$

Since  $t$  was arbitrary, this proves that  $\lim_{k \rightarrow \infty} S_k = 0$ .

10. Suppose the given set contains three non-collinear points, say  $A, B$  and  $C$ , such that  $|AB| = r$  and  $|AC| = s$  are integers. If  $P$  is any point at integral distance from both  $A$  and  $B$ , then by the triangle inequality  $|PA - PB|$  is one of the integers  $0, 1, \dots, r$ . Hence  $P$  must lie on one of the hyperbolas

$$H_j := \{X : |XA - XB| = j\}, \quad j = 1, \dots, r-1,$$

on the line  $H_r$  through the points  $A$  and  $B$  outside the segment  $AB$  itself, or on the perpendicular bisector  $H_0$  of  $AB$ . Analogously, a point  $P$  an integral distance from both  $A$  and  $C$  must be on one of the hyperbolas

$$K_j := \{X : |XA - XC| = j\}, \quad j = 1, \dots, s-1,$$

on the line  $K_s$  through  $A$  and  $C$  or on the perpendicular bisector  $K_0$  of  $AC$ . Any point in our set must be in one of the sets  $H_i \cap K_j$ . Since the lines  $AB$  and  $AC$  are different, none of the sets  $H_i$  coincides with a set  $K_j$ . Thus, for all  $i$  and  $j$ ,  $H_i \cap K_j$  is the intersection of two distinct curves of degree at most two, so it contains at most 4 points. Therefore the given set contains at most  $4(r+1)(s+1)$  points. Contradiction! So, all the points must lie on the same line.

11. Suppose not, i.e.,  $a_{m,n} \leq mn$  for all  $m, n \geq 1$ . Let  $R(k) := \{(i, j) : a_{i,j} \leq k\}$ . By hypothesis,  $\#R(k) \leq 8k$ . On the other hand,  $R(k)$  contains all pairs  $(i, j)$  with  $ij \leq k$ , and there are

$$\left\lfloor \frac{k}{1} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor + \dots + \left\lfloor \frac{k}{k} \right\rfloor > \left(\frac{k}{1} - 1\right) + \left(\frac{k}{2} - 1\right) + \dots + \left(\frac{k}{k} - 1\right) > k(\ln k - 1)$$

such pairs. Hence  $8k > k(\ln k - 1)$ , which is a contradiction.

12. It suffices to show that a triple  $(a, b, c)$  with  $0 < a \leq b \leq c$  can be transformed into a triple  $(b', r, c')$  where  $r$  is the remainder when  $b$  is divided by  $a$ . Hence any set with no zeros can be transformed into a set with a smaller minimal element.

Let  $b = qa + r$  with  $0 \leq r < a$  the remainder and  $q$  the quotient. Let  $q = q_0 + 2q_1 + 4q_2 + \cdots + 2^k q_k$  be the binary representation of  $q$  (so  $q_j \in \{0, 1\}$ , and  $q_k = 1$ ). Define  $g_0(d, e, f) := (2d, e, f - d)$  and  $g_1(d, e, f) := (2d, e - d, f)$ . Then

$$g_{q_k}(g_{q_{k-1}}(\cdots(g_{q_1}(g_{q_0}(a, b, c)))\cdots)) = (b', r, c')$$

is a sequence of allowed moves that produces the desired effect.