Bits and pieces of the nonnegative inverse eigenvalue problem

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Outline

- Nonnegative inverse eigenvalue problem (NIEP)

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- Preserving subclasses of nonnegative matrices
- Related problems
- Open problems

NIEP

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$$\Lambda := (\lambda_1, \lambda_2, \ldots, \lambda_n),$$

does there exist a nonnegative matrix A with $\sigma(A) = \Lambda$?

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Contributions by:

Perron, Frobenius, Wielandt, Kellogg, Dmitriev, Dynkin, Suleimanova, Boyle, Handelman, Friedland, Laffey, Schneider, Neumann, Johnson, Chu, London, Loewy, and many more

Necessary conditions

- $\overline{\Lambda} = \Lambda$, conjugation
- $\max |\Lambda| \in \Lambda$, spectral radius
- $s_k(\Lambda) := \sum_{j=1}^{n} \lambda_j^k \geq 0,\ k \in \mathbb{N}$. moments

Moments $\Longrightarrow$ conjugation & spectral radius conditions [Friedland, 1978].

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Boyle & Handelman [1991]:

Any n-tuple Λ satisfying basic conditions (= nonnegativity of moments) can be augmented by sufficiently many zeros so that the resulting n-tuple $\widetilde{\Lambda} = (\Lambda, 0)$ will be realizable.

Connection to M-matrices

- sign pattern

$$\begin{bmatrix} + & - & - & \cdots & - \\ - & + & - & \cdots & - \\ - & - & + & \cdots & - \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ - & - & - & \cdots & + \end{bmatrix}$$

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- nonnegativity of principal minors

$$A[\alpha] \geq 0, \qquad \text{all } \alpha$$

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- If A is a nonsingular M-matrix, then A^{-1} is nonnegative.

  Open problem: characterize inverses of M-matrices.

NIEP: further conditions

- Johnson-Loewy-London inequalities [1978/79]

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- Newton's inequalities.
  Λ is the spectrum of a nonnegative matrix $\Longleftrightarrow$

$$\mathcal{M} := r(1, 1, \ldots, 1) - \Lambda, \quad r \ge \max |\Lambda|,$$

  is the spectrum of an M-matrix. Then

$$c_k^2(\mathcal{M}) \ge c_{k-1}(\mathcal{M}) c_{k+1}(\mathcal{M}), \quad k = 1, \ldots, n,$$

  where $c_k(\mathcal{M})$ are normalized coefficients of the polynomial with roots $\mathcal{M}$.

Newton’s inequalities

Theorem. Let A be similar to an M- or inverse M-matrix. Then the normalized coefficients

$$c_j(A) := \sum_{\#\alpha=j} A[\alpha] / \binom{n}{j}, \quad j = 0, \ldots, n,$$

of its characteristic polynomial satisfy Newton’s inequalities

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These are spectral conditions.

More on Newton's inequalities

Corollary. If A is similar to an M- or an inverse M-matrix, then

$$c_j(A)^{1/j} \geq c_k(A)^{1/k} \text{ whenever } j \leq k.$$

In particular,

$$c_1(A) \geq c_j(A)^{1/j} \quad \text{for } \ j \geq 1.$$

Main question

For Λ to be a solution to NIEP, it must satisfy moments, JLL, and Newton. Suppose f is an entire function mapping nonnegative matrices of order n into themselves:

$$A \geq 0 \Longrightarrow f(A) \geq 0.$$

Then $f(\Lambda)$ must satisfy moments, JLL, and Newton. Thus we obtain many new inequalities necessary to solve NIEP.

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Question: Characterize all entire functions that leave invariant the cone of nonnegative matrices of order n.

Preliminary observations

Denote the class of entire functions preserving nonnegativity of matrices of order n by $\mathcal{F}_n$ and its restriction to polynomials by $\mathcal{P}_n$.

- $\mathcal{F}_n$ contains all functions with nonnegative Taylor coefficients;
- $\mathcal{F}_n$ is closed under addition, multiplication, and composition;
- $\mathcal{P}_n$ is a union of proper cones obtained by restricting to polynomials of degree bounded by a fixed positive integer. The extreme directions of these cones may be of interest.

Easier questions

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- C_n: nonnegative circulant matrices;
- U_n (L_n): nonnegative upper/lower triangular matrices;
- BU_n (BL_n): nonnegative block upper/lower triangular matrices.

Preserving the class ND_n

All continuous functions that leave ND_n invariant were characterized by Micchelli and Willoughby.

Result [Micchelli & Willoughby 1979]. A function f continuous on $\mathbb{R}_+$ leaves invariant the class ND_n of nonnegative definite entrywise nonnegative symmetric matrices of order n iff all the divided differences of f of order up to n are nonnegative over $\mathbb{R}_+$:

$$f[x_1, \ldots, x_k] \geq 0 \quad x_1, \ldots, x_k \geq 0, \quad k = 1, \ldots, n.$$

Remark on the MW result

The result is not strong enough to characterize matrices preserving the class S_n.

Example.

$$f(x) = 1 + x + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4$$

satisfies the Micchelli-Willoughby conditions of order 2, but it maps the matrix

$$\begin{bmatrix} 0 & r \\ r & 0 \end{bmatrix}, \quad r \text{ sufficiently large,}$$

to a matrix with negative elements.

Preserving the class U_n/L_n

Theorem. A function f continuous on $\mathbb{R}_+$ leaves invariant the class U_n (L_n) of nonnegative upper- (lower-) triangular matrices of order n iff all the divided differences of f of order up to n are nonnegative over $\mathbb{R}_+$.

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Result [Schmitt 1979, Stafney 1978].

$$f(A)_{ij} = \begin{cases} f(a_{ii}) \\ \sum a_{ii_1} \cdots a_{i_k j} f[a_{ii}, a_{i_1 i_1}, \ldots, a_{i_k i_k}, a_{jj}] \\ 0 \end{cases}$$

Preserving the class C_n

Theorem. An entire function f maps the set C_n of nonnegative circulant matrices of order n into itself iff

$$\sum_{k=0}^{n-1} \omega^{-lk} f\left(\sum_{j=0}^{n-1} \omega^{jk} a_j\right) \geq 0 \quad \text{whenever } a_j \geq 0,$$

$$j = 0, \ldots, n-1, \quad \omega := e^{2\pi \mathrm{i}/n}.$$

Proof: is based on an explicit formula connecting eigenvalues of a circulant matrix to its elements.

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Result. The eigenvalues of a circulant matrix with the first row $(a_0, \ldots, a_{n-1})$ are given by

$$\sum_{k=0}^{n-1} a_k \omega^{lk}, \quad l = 0, \ldots, n-1.$$

Further results

- Preserving nonnegative block upper/lower triangular matrices (BU_n/BL_n).

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- Complete characterization for small values of n.

$$n = 2 \qquad \begin{aligned} f(x+y) - f(x-y) &\geq 0, \\ x, y &\geq 0 \\ (x+y)f(x-y) + (y-x)f(x+y) &\geq 0, \\ y \geq x &\geq 0. \end{aligned}$$

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Theorem. An even function $x \mapsto f(x^2)$ leaves S_n invariant iff all the divided differences of f of order up to n are nonnegative over $\mathbb{R}_+$:

$$f[x_1, \ldots, x_k] \geq 0 \quad x_1, \ldots, x_k \geq 0, \quad k = 1, \ldots, n.$$

The same condition is necessary and sufficient for an odd function $x \mapsto x f(x^2)$ to leave S_n invariant.

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The class of functions preserving S_n is not the sum of its even and odd subclasses.

Anti-bidiagonal matrix NIEP

Theorem. A real n-tuple Λ can be realized as the spectrum of a symmetric anti-bidiagonal matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_n \\ 0 & 0 & \cdots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & . & \vdots & \vdots \\ 0 & a_{n-2} & \cdots & 0 & 0 \\ a_n & a_{n-1} & \cdots & 0 & 0 \end{bmatrix}, \quad a_1, \ldots, a_n > 0,$$

iff $\Lambda = (\lambda_1, \ldots, \lambda_n)$ where

$$\lambda_1 > -\lambda_2 > \lambda_3 > \cdots > (-1)^{n-1}\lambda_n > 0.$$

The realizing matrix is necessarily unique.

Sketch of proof

$\Longrightarrow$ An anti-bidiagonal matrix with two positive anti-diagonals is sign-regular with signature sequence

$$1, -1, -1, 1, 1, \cdots, (-1)^{\lceil n-1/2 \rceil}.$$

Invoke a result by Gantmacher and Krein.

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Invoke a result by Gantmacher and Krein.
$\Longleftarrow$ Reduce to the same problem for Jacobi matrices

$$\begin{bmatrix} a_1 & a_2 & \cdots & 0 & 0 \\ a_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_n \\ 0 & 0 & \cdots & a_n & 0 \end{bmatrix}.$$

Sketch of proof

"Reverse" the 3-term recurrence relation to arrive at

$$q_n(\lambda) = (\lambda - a_1)q_{n-1}(\lambda) - a_2^2 q_{n-2}(\lambda),$$
$$q_{n-j}(\lambda) = \lambda q_{n-j-1}(\lambda) - a_{j+2}^2 q_{n-j-2}(\lambda),$$
$$j = 1, \ldots, n-2,$$
$$q_0(\lambda) = 1, \qquad q_1(\lambda) = \lambda.$$

With q_n given, find q_{n-1} and q_{n-2} from parity considerations. Prove root interlacing for q_n and q_{n-1}. The rest is easy (using interlacing repeatedly).

Application

Corollary. Let $\mathcal{M}$ be a real positive n-tuple. Then there exists a Jacobi matrix that realizes $\mathcal{M}$ as its spectrum and has a symmetric anti-bidiagonal square root of the form

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_n \\ 0 & 0 & \cdots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \cdot & \vdots & \vdots \\ 0 & a_{n-2} & \cdots & 0 & 0 \\ a_n & a_{n-1} & \cdots & 0 & 0 \end{bmatrix}, \quad a_1, \ldots, a_n > 0.$$

Open problems

- Can Newton's inequalities be used to prove inequalities conjectured by Boyle & Handelman and Goldberger & Neumann?

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- Can Newton's inequalities be used to prove inequalities conjectured by Boyle & Handelman and Goldberger & Neumann?
- Can Newton's inequalities be used to characterize nonnegative matrices that are inverses of M-matrices?
- What about functions preserving other subclasses of nonnegative matrices, e.g., Toeplitz or Hankel?
- Can the cone theory and the semigroup theory be used to characterize functions that preserve nonnegativity of matrices?

Papers

- *M-matrices satisfy Newton's inequalities* [Proceedings of AMS 2005]
- *The inverse eigenvalue problem for symmetric anti-bidiagonal matrices* [LAA 200?]
- *Functions preserving nonnegativity of matrices*, with G. Bharali [coming soon]

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Thanks

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