



Hyperdeterminantal relations among symmetric principal minors

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joint work with Bernd Sturmfels

Goal and problem

Given a matrix A , let A_I denote its **principal minor** with rows and columns indexed by I .

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Example : $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$

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$$A_{1,3} = \det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

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We give an answer in the symmetric case.

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- **combinatorics:** counting spanning trees of a graph;
- **probability & math. physics:** determinantal processes (quantum mechanics of fermions, eigenvalues of random matrices, random spanning trees and non-intersecting paths).

Hyperdeterminantal relations

$2 \times 2 \times 2$ -hyperdeterminant of A_* must vanish:

$$\begin{aligned} & A_{\emptyset}^2 A_{123}^2 + A_1^2 A_{23}^2 + A_2^2 A_{13}^2 + A_3^2 A_{12}^2 \\ & + 4 \cdot A_{\emptyset} A_{12} A_{13} A_{23} + 4 \cdot A_1 A_2 A_3 A_{123} \\ & - 2 \cdot A_{\emptyset} A_1 A_{23} A_{123} - 2 \cdot A_{\emptyset} A_2 A_{13} A_{123} \\ & - 2 \cdot A_{\emptyset} A_3 A_{12} A_{123} - 2 \cdot A_1 A_2 A_{13} A_{23} \\ & - 2 \cdot A_1 A_3 A_{12} A_{23} - 2 \cdot A_2 A_3 A_{12} A_{13} = 0. \end{aligned}$$

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This comes from

$$\begin{aligned} & (A_{123} - A_{12}A_3 - A_{13}A_2 - A_{23}A_1 + 2A_1A_2A_3)^2 \\ & = 4 \cdot (A_1A_2 - A_{12})(A_2A_3 - A_{23})(A_1A_3 - A_{13}). \end{aligned}$$

Moreover, the $2 \times 2 \times 2$ hyperdeterminantal relations hold after the substitution

$$\begin{aligned} A_{\emptyset} &\mapsto A_I, & A_1 &\mapsto A_{I \cup \{j_1\}} \\ A_2 &\mapsto A_{I \cup \{j_2\}}, & A_3 &\mapsto A_{I \cup \{j_3\}}, \\ A_{12} &\mapsto A_{I \cup \{j_1, j_2\}}, & A_{13} &\mapsto A_{I \cup \{j_1, j_3\}}, \\ A_{23} &\mapsto A_{I \cup \{j_2, j_3\}}, & A_{123} &\mapsto A_{I \cup \{j_1, j_2, j_3\}} \end{aligned}$$

for any subset $I \subset [n]$ and any $j_1, j_2, j_3 \in [n] \setminus I$.

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We call all of these **hyperdeterminantal relations of format $2 \times 2 \times 2$** .

Schur complements

The **Schur complement** of a principal submatrix H in a matrix A is the matrix $A/H := E - FH^{-1}G$

$$\text{where } A =: \begin{bmatrix} E & F \\ G & H \end{bmatrix}.$$

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Schur's identity

$$(A/H)_\alpha = \frac{A_{I \cup \alpha}}{A_I},$$

holds, assuming H is indexed by I .

Consequences of Schur

Any identity among principal minors can be generalized using Schur complements:

- replace each minor A_α by $A_{\alpha \cup I} / A_I$,
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Any identity among principal minors can be generalized using Schur complements:

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Theorem. The principal minors of a symmetric matrix satisfy hyperdeterminantal relations of format $2 \times 2 \times 2$.

Converse?

Theorem. Let A_* be a real vector of length 2^n with $A_\emptyset = 1$ that satisfies

$$A_{I \cap J} A_{I \cup J} < A_I A_J \text{ whenever } \#I \cap J = \#I - 1 = \#J - 1.$$

There exists a real symmetric matrix A with principal minors given by A_* iff A_* satisfies the $2 \times 2 \times 2$ hyperdeterminantal relations and certain additional relations.

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There exists a real symmetric matrix A with principal minors given by A_* iff A_* satisfies the $2 \times 2 \times 2$ hyperdeterminantal relations and certain additional relations.

Corollary. Let $A_* \in \mathbb{R}^{2^n}$ be a vector that can be approximated arbitrarily well by vectors of principal minors of real symmetric matrices. Then there exists a real symmetric matrix A that realizes A_* .

Problem restated

From the point of view of algebraic geometry, the following version of our problem is most natural:

Problem. Let P_n be the prime ideal of all homogeneous polynomial relations among the principal minors of a symmetric $n \times n$ -matrix. Determine a finite set of generators for P_n .

Problem restated

From the point of view of algebraic geometry, the following version of our problem is most natural:

Problem. Let P_n be the prime ideal of all homogeneous polynomial relations among the principal minors of a symmetric $n \times n$ -matrix. Determine a finite set of generators for P_n .

Theorem. The homogeneous prime ideal P_4 is minimally generated by twenty quartics in the 16 unknowns A_I . The corresponding irreducible variety in P^{15} has codimension five and degree 96.

Big hyperdeterminants

The **hyperdeterminant** of an $n_1 \times \cdots \times n_r$ tensor $\mathbf{A} = (\mathbf{a}_{i_1, \dots, i_r})$ is defined from the multilinear form f :

$$f(x) := f(x^{(1)}, \dots, x^{(r)}) := \sum_{i_1, \dots, i_r} \mathbf{a}_{i_1, \dots, i_r} x_{i_1}^{(1)} \cdots x_{i_r}^{(r)}.$$

The hyperdeterminant $\det(\mathbf{A}) = 0$ if and only if the equations

$$f(x) = \frac{\partial f(x)}{\partial x_i^{(j)}} = 0 \quad \text{for all } i, j$$

have a solution $x = (x^{(1)}, \dots, x^{(r)})$ where each $x^{(j)}$ is a non-zero complex vector.

Big hyperdeterminants

Theorem. Let A be a symmetric $n \times n$ matrix. Then the tensor A_* of all principal minors of A is a common zero of all the hyperdeterminants of formats up to

$$\underbrace{2 \times 2 \times \cdots \times 2.}_{n \text{ terms}}$$

n terms

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Thanks!