Open problems on GKK τ-matrices

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Abstract

We propose several open problems on GKK τ-matrices raised by examples showing that some such matrices are unstable.

1 Motivation

A complex square matrix is called '(positive) stable' if all its eigenvalues have positive real part. Such stability of a matrix, or more usually of its negative, is a basic concern in several fields of mathematics, e.g., in the study of ODEs. Three important classes of matrices, namely the nonsingular totally nonnegative matrices, the Hermitian positive definite matrices, and the M-matrices are known to be stable. Matrices in these three classes also share other properties: they are P-matrices, they are weakly sign symmetric, and they satisfy eigenvalue monotonicity (for definitions of these terms see below). In order to investigate the relations between these properties supersets of the three basic matrix classes defined in terms of some or all of the above properties have been studied. Thus Gantmacher-Krein [7], Kotelyansky [10], Fan [6] and Carlson [2], [3] investigated properties of P-matrices that are weakly sign symmetric (the so called GKK matrices), while Engel-Schneider [5] considered τ-matrices which are defined as P-matrices that satisfy eigenvalue monotonicity. The latter were further investigated by Varga [14], Hershkowitz-Berman [9] and Mehrmann [12]. The stability of GKK and τ-matrices was conjectured and proved in low dimensions in various papers, see Holtz [11] for details and further history and see also the survey by Hershkowitz [8].

The GKK τ-matrices satisfy the three properties discussed above. Nevertheless, Holtz [11] has shown that there are unstable GKK τ-matrices. This raises the question under what conditions a GKK τ-matrix is stable. It is the purpose of this note to outline five problems related to this question.

2 Basic notions

Let $\langle n\rangle$ denote the collection of all increasing sequences with elements from the set $\{1, 2, \ldots, n\}$ and let $\#\alpha$ denote the size of the sequence α. Given a matrix $A \in \mathbb{C}^{n\times n}$, we will use the notation $A(\alpha, \beta)$ for the submatrix of A whose rows are indexed by α and columns by β (α, $\beta \in \langle n\rangle$) and $A[\alpha, \beta]$ for the minor $\det A(\alpha, \beta)$ if $\#\alpha = \#\beta$. For simplicity, $A(\alpha)$ will stand for $A(\alpha, \alpha)$ and $A[\alpha]$ for $A[\alpha, \alpha]$. By definition, $A[\emptyset] := 1$. Given $\alpha, \beta \in \langle n\rangle$ with $\#\alpha = \#\beta$, call the number $\#\alpha - \#(\alpha \cap \beta)$ the *dispersal* of the minor $A[\alpha, \beta]$. A matrix A is called a *P-matrix* if $A[\alpha] > 0$ for all $\alpha \in \langle n\rangle$. A is said to be *sign-symmetric* if

$$A[\alpha, \beta]A[\beta, \alpha] \geq 0 \tag{1}$$

for all minors $A[\alpha, \beta]$. A is called *weakly sign-symmetric* [7] if (1) holds for all minors $A[\alpha, \beta]$ with dispersal 1, and these are also referred to as *almost principal*. Weakly sign-symmetric P-matrices were called *GKK* in [6] after Gantmacher, Krein, and Kotelyansky. A P-matrix is *strict GKK* if the inequalities in (1) are strict for all almost principal minors. Let $\sigma(A)$ denote the spectrum of A and let

$$l(A) := \min \sigma(A) \cap \mathbb{R},$$

with the understanding that, in this setting, $\min \emptyset = \infty$. A matrix A is called an *ω-matrix* [5] if it has eigenvalue monotonicity in the sense that

$$l(A(\alpha, \alpha)) \leq l(A(\beta, \beta)) < \infty \qquad \text{whenever} \quad \emptyset \neq \beta \subseteq \alpha \in \langle n\rangle.$$

A is a *τ-matrix* if, in addition, $l(A) \geq 0$. A matrix is called *positive stable* if its spectrum lies entirely in the open right half plane. In the sequel, we will shorten the term 'positive stable' to simply 'stable'.

3 GKK τ-matrix stability and related problems

1. **Strict GKK matrices: closure of the set.** What GKK matrices can be approximated arbitrarily well by strict GKK matrices? In particular, can the matrices constructed in [11] be approximated by τ-matrices that are strict GKK? The matrices in [11] themselves are not strict GKK. The negative answer to this question would give rise to Problem 1a.

   1a. Strict GKK matrices: stability. Are strict GKK matrices stable? Are strict GKK τ-matrices stable?

2. **Dispersal condition sufficient for stability.** The counterexample given in [11] shows that it is not sufficient for stability of a P-matrix A that the inequalities (1) hold for all minors of dispersal less than or equal to $d = 1$. Carlson's theorem [3] asserts that the value $d = n$ is sufficient for stability (in other words, that sign-symmetric matrices are stable). What minimal value of the parameter d would guarantee stability? In particular, does that value depend on n?

3. **Classes of stable matrices.** Following a conjecture by Varga [14], Mehrmann [12] showed that τ-matrices of order up to 4 satisfy the inequality

$$|\arg(\lambda - l(A))| \leq \frac{\pi}{2} - \frac{\pi}{n} \qquad \forall \lambda \in \sigma(A), \tag{2}$$

which is a property stronger than stability.

However, in [11] it was shown that the GKK τ-matrices $A_{2k+2,k,t}$ of order $2k+2$ defined there are unstable for all integers $k,\ k > 20$, and all sufficiently small positive t (and numerical evidence suggests that there may be matrices $A_{n,k,t}$ of smaller order that are unstable). For various classes $\mathcal{C}$ of matrices, this raises the following three questions concerning the subclass $\mathcal{C}_n$ of all matrices in $\mathcal{C}$ of order less than or equal to n.

3a. What is the maximum n such that all matrices in class $\mathcal{C}_n$ are stable?

3b. What is the maximum n such that all matrices in class $\mathcal{C}_n$ satisfy (2)?

3c. For given n, do the stable matrices in class $\mathcal{C}_n$ satisfy (2)?

Specific classes $\mathcal{C}$ of interest include

i. The class of matrices $A_{n,k,t}$ for positive integers n and k, and for $t \in (0,1)$,

ii. The class of GKK τ-matrices,

iii. The class of GKK matrices,

iv. The class of τ-matrices.

4. **Assignment of principal minors.** Given $n \in \mathbb{N}$ and numbers $(p_\alpha)_{\emptyset \neq \alpha \in \langle n \rangle}$, is there a matrix A such that $A[\alpha] = p_\alpha$ for all α?

This question was originally motivated by the Gantmacher-Krein-Carlson theorem ([7] and [2]), which states that a P-matrix A is GKK if and only if its minors satisfy the generalized Hadamard-Fischer inequality

$$A[\alpha]A[\beta] \geq A[\alpha \cup \beta]A[\alpha \cap \beta] \qquad \forall \alpha, \beta \in \langle n \rangle. \tag{3}$$

An answer to Problem 4, coupled with inequalities (3), would therefore allow to decide, for a collection of positive numbers (p_α), whether there exists a GKK matrix such that $A[\alpha] = p_\alpha$ for all α. Since $\sigma(A)$ is determined by the numbers p_α, one could then find, at least in principle, all possible spectra of GKK matrices.

This problem is in fact equivalent to a certain inverse eigenvalue problem. Indeed, specifying all principal minors implies specifying characteristic polynomials, and hence all eigenvalues, of the principal submatrices, and vice versa.

Problem 4 can be specialized to various classes of matrices, e.g., Hermitian or nonnegative. A part of this problem for Hermitian matrices is solved in [1], where it is shown how to construct a symmetric m-band matrix of order n from the given (necessarily real) eigenvalues of the m leading principal submatrices of greatest order. Such a matrix always exists (but is not unique) whenever the eigenvalues of $A(1{:}j)$ interlace those of $A(1{:}j{+}1)$, $j = n-m+1, \ldots, n-1$. Notice that, by the Hermite-Biehler theorem (see, e.g., [4, p.21]), two real polynomials p and q have real interlacing roots if and only if the roots of the polynomial $p + iq$ are all on the same side of the real axis, and the latter can be checked using the Hurwitz matrix for the polynomial $w(z) := p(iz) + iq(iz)$, which is constructed from the coefficients of p and q. Therefore, the interlacing property can be checked using only the coefficients of characteristic polynomials of principal submatrices, and these are sums of the given numbers p_α.

5. Newton's inequalities. For a matrix A, let

$$c_j := \sum_{\#\alpha=j} A[\alpha]/\binom{n}{j}, \qquad j = 0, \dots, n.$$

Does any GKK τ-matrix A satisfy the inequalities

$$c_j^2 \geq c_{j-1}c_{j+1}, \quad j = 1, \dots, n-1? \tag{4}$$

This problem has possible subproblems, e.g.,

5a. Newton's inequalities for M-matrices.

5b. Newton's inequalities for stable GKK τ-matrices.

These inequalities are known for real diagonal matrices, i.e., simply for sequences of real numbers (see [13] and references therein), as was first proved by Newton. Since the numbers c_j are invariant under similarity, Newton's inequalities (4) also hold for all diagonalizable matrices with real spectrum, and therefore also for the closure of this set, viz. for *all* matrices with real spectrum.

This question arose at an early stage of the work that led to [11] in an attempt to prove that Newton's inequalities hold for the GKK matrices and imply their stability. When it became clear that there exist P-matrices satisfying Newton's inequalities that are unstable, the author of [11] turned to constructing a counterexample to the GKK matrix stability conjecture. Whether GKK matrices satisfy Newton's inequalities, however, is not known yet.

We hope that consideration of our questions will lead to further interesting results on GKK τ-matrices.

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