

# Classification of normal operators in spaces with indefinite scalar product of rank 2

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## Abstract

A finite-dimensional complex space with indefinite scalar product  $[\cdot, \cdot]$  having  $v_- = 2$  negative squares and  $v_+ \geq 2$  positive ones is considered. The paper presents a classification of operators that are normal with respect to this product. It relates to the paper [1], where the similar classification was obtained by Gohberg and Reichstein for the case  $v = \min\{v_-, v_+\} = 1$ .

## 1 Introduction

Consider a complex linear space  $C^n$  with an indefinite scalar product  $[\cdot, \cdot]$ . By definition, the latter is a nondegenerate sesquilinear Hermitian form. If the ordinary scalar product  $(\cdot, \cdot)$  is fixed, then there exists a nondegenerate Hermitian operator  $H$  such that  $[x, y] = (Hx, y) \quad \forall x, y \in C^n$ . If  $A$  is a linear operator ( $A : C^n \rightarrow C^n$ ), then the  $H$ -adjoint of  $A$  (denoted by  $A^{[*]}$ ) is defined by the identity  $[A^{[*]}x, y] = [x, Ay]$  (hence  $A^{[*]} = H^{-1}A^*H$ ). An operator  $N$  is called  $H$ -normal if  $NN^{[*]} = N^{[*]}N$ , an operator  $U$  is called  $H$ -unitary if  $UU^{[*]} = I$ , where  $I$  is the identity transformation.

Let  $V$  be a nontrivial subspace of  $C^n$ .  $V$  is called *neutral* if  $[x, y] = 0$  for all  $x, y \in V$ . In this case we may write  $[V, V] = 0$ .  $V$  is called *nondegenerate* if from  $x \in V$  and  $\forall y \in V [x, y] = 0$  it follows that  $x = 0$ . The subspace  $V^{[\perp]}$  is defined as the set of all vectors  $x \in C^n$ :  $[x, y] = 0 \quad \forall y \in V$ . If  $V$  is nondegenerate, then  $V^{[\perp]}$  is also nondegenerate and  $V \dot{+} V^{[\perp]} = C^n$ .

A linear operator  $A$  acting in  $C^n$  is called *decomposable* if there exists a nondegenerate subspace  $V \subset C^n$  such that both  $V$  and  $V^{[\perp]}$  are invariant for  $A$ . Then  $A$  is the *orthogonal sum* of  $A_1 = A|_V$  and  $A_2 = A|_{V^{[\perp]}}$ . Since the conditions  $AV^{[\perp]} \subseteq V^{[\perp]}$  and  $A^{[*]}V \subseteq V$  are equivalent, an operator  $A$  is decomposable if there exists a nondegenerate subspace  $V$  which is invariant both for  $A$  and  $A^{[*]}$ .

Pairs of matrices  $\{A_1, H_1\}$  and  $\{A_2, H_2\}$ , where  $H_1$  and  $H_2$  are Hermitian, are called *unitarily similar* if  $A_2 = T^{-1}A_1T$ ,  $H_2 = T^*H_1T$  for some invertible  $T$ ; in case when  $H_1 = H_2$  they are  $H_1$ -unitarily similar.

Throughout what follows by a rank of a space we mean  $v = \min\{v_-, v_+\}$ , where  $v_-$  ( $v_+$ ) is the number of negative (positive) squares of the quadratic form  $[x, x]$ , or (it is the same) the number of negative (positive) eigenvalues of the operator  $H$ . Note that without loss of generality it can be assumed that  $v_- \leq v_+$  (otherwise  $H$  can be replaced by  $-H$ ; the latter (invertible and Hermitian operator) has opposite eigenvalues).

Our aim is to obtain a complete classification for  $H$ -normal operators acting in the space  $C^n$  of rank 2, i.e., to find a set of canonical forms such that any  $H$ -normal operator could be reduced to one and only one of these forms. This means that for any invertible Hermitian matrix  $H$  with  $v = 2$  and for any  $H$ -normal matrix  $N$  we must point out one and only one of the canonical pairs of matrices  $\{\tilde{N}, \tilde{H}\}$  such that the pair  $\{N, H\}$  is unitarily similar to  $\{\tilde{N}, \tilde{H}\}$ .

Since any  $H$ -normal operator  $N : C^n \rightarrow C^n$  is an orthogonal sum of  $H$ -normal operators each of which has one or two distinct eigenvalues (Lemma 1 from [1]), it is sufficient to solve our problem only for indecomposable operators having one or two distinct eigenvalues.

Thus, in this paper we consider only indecomposable operators having one or two distinct eigenvalues and assume that  $2 = v_- \leq v_+$ .

Finally let us introduce some notation. Denote the identity matrix of order  $r \times r$  by  $I_r$ , the  $r \times r$  matrix with 1's on the secondary diagonal and zeros elsewhere by  $D_r$ , and a block diagonal matrix with  $A, B, \dots, C$  diagonal blocks by  $A \oplus B \oplus \dots \oplus C$ :

$$I_r = \begin{pmatrix} 1 & & 0 \\ & \cdot & \\ 0 & & 1 \end{pmatrix}, \quad D_r = \begin{pmatrix} 0 & & 1 \\ & \cdot & \\ 1 & & 0 \end{pmatrix},$$

$$A \oplus B \oplus \dots \oplus C = \begin{pmatrix} A & & 0 \\ & B & \\ 0 & & C \end{pmatrix}.$$

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## 2 Some Properties of Indecomposable $H$ -normal Operators

The results of this section hold for any finite-dimensional space with indefinite scalar product.

**Proposition 1** *Let an indecomposable  $H$ -normal operator  $N$  acting in  $C^n$  ( $n > 1$ ) have the only eigenvalue  $\lambda$ ; then there exists a decomposition of  $C^n$  into a direct sum of subspaces*

$$S_0 = \{x \in C^n : (N - \lambda I)x = (N^{[*]} - \bar{\lambda}I)x = 0\}, \quad (1)$$

$S, S_1$  such that

$$N = \begin{pmatrix} N' = \lambda I & * & * \\ 0 & N_1 & * \\ 0 & 0 & N'' = \lambda I \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & I \\ 0 & H_1 & 0 \\ I & 0 & 0 \end{pmatrix}, \quad (2)$$

where  $N' : S_0 \rightarrow S_0$ ,  $N_1 : S \rightarrow S$ ,  $N'' : S_1 \rightarrow S_1$ , the internal operator  $N_1$  is  $H_1$ -normal, and the pair  $\{N_1, H_1\}$  is determined up to the unitary similarity.

**Proof:** Since  $N$  and  $N^{[*]}$  commute, the subspace  $S_0$  defined by (1) is nontrivial. For  $N$  to be indecomposable  $S_0$  must be neutral. Indeed, otherwise  $\exists v \in S_0 : Nv = \lambda v$ ,  $N^{[*]}v = \bar{\lambda}v$ ,  $[v, v] \neq 0$ , therefore,  $V = \text{span}\{v\}$  is a nondegenerate subspace that is invariant both for  $N$  and  $N^{[*]}$ , hence,  $N$  is decomposable. Thus,  $S_0$  is neutral. Let us take advantage of the following well-known result: *for any neutral subspace  $V_1 \subset C^n$  there exists a subspace  $V_2$  ( $V_1 \cap V_2 = \{0\}$ ) such that*

$$H|_{(V_1+V_2)} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \quad (3)$$

Therefore, for  $S_0$  there exists a neutral subspace  $S_1$  such that  $H|_{(S_0+S_1)}$  has form (3). Since the subspace  $(S_0+S_1)$  is nondegenerate, the subspace  $S = (S_0+S_1)^{\perp}$  is also nondegenerate and  $C^n = S_0 \dot{+} S \dot{+} S_1$ . As  $\forall v \in C^n$   $(N - \lambda I)v \in S_0^{\perp}$  and  $(N^{[*]} - \bar{\lambda}I)v \in S_0^{\perp}$ , the matrices  $N$  and  $H$  has form (2) with respect to the decomposition  $C^n = S_0 \dot{+} S \dot{+} S_1$ . Since  $N$  is  $H$ -normal, the internal operator  $N_1$  is  $H_1$ -normal.

It is seen that only the subspace  $S_0$  is fixed;  $S$  and  $S_1$  may change. However, the pair  $\{N_1, H_1\}$  is unique in a sense, namely, it is determined up to the unitary similarity. Indeed, any transformation  $T$  such that  $TS_0 \subseteq S_0$  has the form

$$T = \begin{pmatrix} T_1 & T_2 & T_3 \\ 0 & T_4 & T_5 \\ 0 & T_6 & T_7 \end{pmatrix}.$$

Since

$$\tilde{H} = \begin{pmatrix} 0 & 0 & I \\ 0 & \tilde{H}_1 & 0 \\ I & 0 & 0 \end{pmatrix},$$

from condition  $\tilde{H} = T^*HT$  it follows that  $T_6 = 0$ ,  $\tilde{H}_1 = T_4^*H_1T_4$ . As  $\tilde{N} = T^{-1}NT$ ,  $\tilde{N}_1 = T_4^{-1}N_1T_4$  so that the pair  $\{N_1, H_1\}$  is unitarily similar to  $\{\tilde{N}_1, \tilde{H}_1\}$ , Q.E.D.

**Remark:** the decomposition  $C^n = S_0 \dot{+} S \dot{+} S_1$  was constructed in [1], section 6 so that the first part of this statement is borrowed from [1].

**Corollary:** to go over from one decomposition  $C^n = S_0 \dot{+} S \dot{+} S_1$  to another by means of a transformation  $T$  it is necessary that  $T$  would be block triangular with respect to both decompositions.

**Theorem 1** *If an  $H$ -normal operator  $N$  acting in a space  $C^n$  of rank  $k \geq 1$  is indecomposable, then either (A) or (B) holds:*

(A)  $N$  has two eigenvalues and  $n = 2k$ ;

(B)  $N$  has one eigenvalue and  $2k \leq n \leq 4k$ .

**Proof:** First show that  $n \geq 2k$ . Indeed,  $n = v_- + v_+ \geq 2 \min\{v_-, v_+\} = 2k$ . Now prove (A). Let  $N$  have two distinct eigenvalues. Then, according to Lemma 1 form [1],  $C^n$  is a direct sum of two neutral subspaces of the same dimension  $m$  which are invariant for  $N$  and  $N^{[*]}$ . Since in a space with indefinite scalar product no neutral space can be of dimension more than rank of a space,  $m \leq k$  and  $n \leq 2k$ . But it is established before that  $n \geq 2k$ . Hence,  $n = 2k$  and the proof of (A) is completed.

Now prove (B), i.e., show that if  $N$  has one eigenvalue, then  $n \leq 4k$ . For  $k = 1$  the proof is given in Theorem 1, [1]. Suppose inductively that for all  $i \leq k$  the size of indecomposable operators having one eigenvalue is not more than  $4i \times 4i$ . Let  $v_- = k + 1$ ,  $v_+ \geq v_-$ ,  $N$  have the only eigenvalue  $\lambda$ . According to Proposition 1, one can assume that the matrices  $N$  and  $H$  has form (2). Let  $N_1 = N_1^{(1)} \oplus \dots \oplus N_1^{(p)}$  be a decomposition of the internal operator  $N_1$  into an orthogonal sum of indecomposable operators,  $H_1 = H_1^{(1)} \oplus \dots \oplus H_1^{(p)}$ ,  $S = S^{(1)} \oplus \dots \oplus S^{(p)}$  be the corresponding decompositions of  $H_1$  and  $S$ . Let  $v_-^{(i)}$  be the number of negative eigenvalues of  $H_1^{(i)}$  ( $i = 1, \dots, p$ ). If  $\dim S_0 = s$ , then  $\sum_{i=1}^p v_-^{(i)} = k + 1 - s$ . Let

$$H_1' = \sum_{v_-^{(i)} > 0} H_1^{(i)}, \quad H_1'' = \sum_{v_-^{(i)} = 0} H_1^{(i)}.$$

Then  $H_1 = H_1' \oplus H_1''$ ,  $N_1 = N_1' \oplus N_1''$ , where  $N_1'$ ,  $N_1''$  are the corresponding sums of operators  $N_1^{(i)}$ . Since for any  $i = 1, \dots, p$  rank of the subspace  $S_1^{(i)}$  is not more than  $v_-^{(i)}$ ,  $v_-^{(i)} \leq k$  (because  $k + 1 - s \leq k$ ), and the size of an indecomposable operator in a space of rank 0 is equal to 1, by the inductive hypothesis  $\dim S^{(i)} \leq 4v_-^{(i)}$ , hence  $\dim S' \leq 4(k + 1 - s)$ . Since  $H_1''$  has only positive eigenvalues,  $N_1''$  is a usual normal operator having one eigenvalue  $\lambda$ , therefore,  $N_1'' = \lambda I$  so that

$$N = \begin{pmatrix} \lambda I & * & M_1 & * \\ 0 & N_1' & 0 & * \\ 0 & 0 & \lambda I & * \\ 0 & 0 & 0 & \lambda I \end{pmatrix}, \quad N^{[*]} = \begin{pmatrix} \bar{\lambda} I & * & M_2 & * \\ 0 & N_1'^{[*]} & 0 & * \\ 0 & 0 & \bar{\lambda} I & * \\ 0 & 0 & 0 & \bar{\lambda} I \end{pmatrix}.$$

If  $\dim S'' = r > 2s$ , then the system

$$\begin{aligned} M_1 X &= 0 \\ M_2 X &= 0 \end{aligned}$$

has a nontrivial solution  $X = (x_1, \dots, x_r)^T$  (where  $Y^T$  is  $Y$  transposed). Therefore, there exists a nonzero vector  $v = \sum_{i=1}^r x_i w_i$  ( $w_i$  are the basis vectors of  $S''$ ) that satisfies the condition  $(N - \lambda I)v = (N^{[*]} - \bar{\lambda} I)v = 0$ , i.e.,  $v \in S_0$ . But  $S_0 \cap S = \{0\}$ . This contradiction proves that  $\dim S'' \leq 2s$ . Thus,  $n = 2 \dim S_0 + \dim S' + \dim S'' \leq 2s + 4(k + 1 - s) + 2s = 4(k + 1)$ , Q.E.D.

Since an indecomposable operator cannot have more than two eigenvalues (Lemma 1, [1]), either (A) or (B) is true so that the proof of the theorem is completed.

### 3 The Classification of Indecomposable $H$ -normal Operators

The principal aim of this paper is to prove the following result:

**Theorem 2** *If an indecomposable  $H$ -normal operator  $N$  ( $N : C^n \rightarrow C^n$ ) acts in a space with indefinite scalar product with  $v_- = 2$  negative squares and  $v_+ \geq 2$  positive ones, then  $4 \leq n \leq 8$  and the pair  $\{N, H\}$  is unitarily similar to one and only one of canonical pairs  $\{(4), (5)\} - \{(31), (32)\}$ . The choice of the particular canonical form is determined as follows.*

*If  $N$  has two distinct eigenvalues  $\lambda_1, \lambda_2$ , then  $\{N, H\}$  is unitarily similar to  $\{(4), (5)\}$ :*

$$N = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & x & \lambda_2 \end{pmatrix}, \quad x \in C,$$

$$\text{for } x \neq 0 \begin{cases} \operatorname{Im}\{\lambda_1 - \lambda_2\} > 0 & \text{if } \operatorname{Im}\{\lambda_1 - \lambda_2\} \neq 0, \\ \operatorname{Re}\{\lambda_1 - \lambda_2\} > 0 & \text{otherwise,} \end{cases} \quad (4)$$

$$H = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \quad (5)$$

*If  $N$  has one eigenvalue  $\lambda$ ,  $\dim S_0 = 1$ , the internal operator  $N_1$  is indecomposable, and  $n = 4$ , then  $\{N, H\}$  is unitarily similar to  $\{(6), (7)\}$ :*

$$N = \begin{pmatrix} \lambda & 1 & ir_1 & ir_2z \\ 0 & \lambda & z & 0 \\ 0 & 0 & \lambda & z^2 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad |z| = 1, \quad r_1, r_2 \in \mathfrak{R}, \quad (6)$$

$$H = D_4. \quad (7)$$

*If  $N$  has one eigenvalue  $\lambda$ ,  $\dim S_0 = 1$ ,  $N_1$  is indecomposable, and  $n = 5$ , then  $\{N, H\}$  is unitarily similar to one and only one of pairs  $\{(8), (11)\}, \{(9), (11)\}, \{(10), (11)\}$ :*

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 & ir_3 \\ 0 & \lambda & 1 & ir_1 & -2r_1^2 + ir_2 \\ 0 & 0 & \lambda & 1 & 2ir_1 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \quad r_1, r_2, r_3 \in \mathfrak{R}, \quad (8)$$

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 & ir_3 \\ 0 & \lambda & z & r_1 & -2z^2r_1^2\operatorname{Im}^2z + ir_2z^2 \\ 0 & 0 & \lambda & z & -2ir_1z^2\operatorname{Im}z \\ 0 & 0 & 0 & \lambda & z^2 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \begin{cases} |z| = 1, \quad z \neq i, \\ 0 < \arg z < \pi, \\ r_1, r_2, r_3 \in \mathfrak{R}, \end{cases} \quad (9)$$

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 & r_3 \\ 0 & \lambda & i & r_1 & 2r_1^2 + ir_2 \\ 0 & 0 & \lambda & i & 2ir_1 \\ 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \quad r_1, r_2, r_3 \in \mathfrak{R}, \quad (10)$$

$$H = D_5. \quad (11)$$

*If  $N$  has one eigenvalue  $\lambda$ ,  $\dim S_0 = 1$ ,  $N_1$  is decomposable, and  $n = 4$ , then  $\{N, H\}$  is unitarily similar to one and only one of pairs  $\{(12), (15)\}, \{(13), (15)\}, \{(14), (15)\}$ :*

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & z \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad |z| = 1, \quad (12)$$

$$N = \begin{pmatrix} \lambda & 1 & 1 & 0 \\ 0 & \lambda & 0 & z \\ 0 & 0 & \lambda & (1+ir)z \\ 0 & 0 & 0 & \lambda \end{pmatrix}, |z| = 1, r \in \Re > 0, \quad (13)$$

$$N = \begin{pmatrix} \lambda & 1 & -1 & 0 \\ 0 & \lambda & 0 & z \\ 0 & 0 & \lambda & -(1+ir)z \\ 0 & 0 & 0 & \lambda \end{pmatrix}, |z| = 1, r \in \Re > 0, \quad (14)$$

$$H = D_4. \quad (15)$$

If  $N$  has one eigenvalue  $\lambda$ ,  $\dim S_0 = 1$ ,  $N_1$  is decomposable, and  $n = 5$ , then  $\{N, H\}$  is unitarily similar to  $\{(16), (17)\}$ :

$$N = \begin{pmatrix} \lambda & 1 & 0 & \frac{1}{2}r_1^2 + ir_2 & 0 \\ 0 & \lambda & 0 & z & 0 \\ 0 & 0 & \lambda & 0 & r_1 \\ 0 & 0 & 0 & \lambda & z^2 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, |z| = 1, r_1, r_2 \in \Re, r_1 > 0, \quad (16)$$

$$H = D_5. \quad (17)$$

If  $N$  has one eigenvalue  $\lambda$ ,  $\dim S_0 = 1$ ,  $N_1$  is decomposable, and  $n = 6$ , then  $\{N, H\}$  is unitarily similar to one and only one of pairs  $\{(18), (20)\}$ ,  $\{(19), (20)\}$ :

$$N = \begin{pmatrix} \lambda & 1 & 2ir_1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & ir_1 & 0 & 2r_1^2 - r_2^2/2 + ir_3 \\ 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 1 \\ 0 & 0 & 0 & 0 & \lambda & r_2 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, r_1, r_2 \in \Re, r_2 > 0, \quad (18)$$

$$N = \begin{pmatrix} \lambda & 1 & -2ir_1 \Im z & 0 & 0 & 0 \\ 0 & \lambda & z & r_1 & 0 & (2r_1^2 \Im^2 z - r_2^2/2 + ir_3)z^2 \\ 0 & 0 & \lambda & z & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & z^2 \\ 0 & 0 & 0 & 0 & \lambda & r_2 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$|z| = 1, 0 < \arg z < \pi, r_1, r_2, r_3 \in \Re, r_2 > 0, \quad (19)$$

$$H = \begin{pmatrix} 0 & 0 & 0 & I_1 \\ 0 & D_3 & 0 & 0 \\ 0 & 0 & I_1 & 0 \\ I_1 & 0 & 0 & 0 \end{pmatrix}. \quad (20)$$

If  $N$  has one eigenvalue  $\lambda$ ,  $\dim S_0 = 2$ , and  $n = 4$ , then  $\{N, H\}$  is unitarily similar to one and only one of pairs  $\{(21), (23)\}$ ,  $\{(22), (23)\}$ :

$$N = \begin{pmatrix} \lambda & 0 & z & re^{-i\pi/3}z \\ 0 & \lambda & 0 & e^{i\pi/3}z \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \begin{array}{l} |z| = 1, r \in \Re \geq \sqrt{3}, \\ 0 \leq \arg z < \pi \text{ if } r > \sqrt{3}, \end{array} \quad (21)$$

$$N = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad (22)$$

$$H = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \quad (23)$$

If  $N$  has one eigenvalue  $\lambda$ ,  $\dim S_0 = 2$ , and  $n = 5$ , then  $\{N, H\}$  is unitarily similar to one and only one of pairs  $\{(24), (26)\}$ ,  $\{(25), (26)\}$ :

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 \\ 0 & 0 & \lambda & z & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \quad |z| = 1, \quad (24)$$

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & r & z \\ 0 & 0 & \lambda & z^2 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \quad |z| = 1, \quad r \in \Re > 0, \quad (25)$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_1 & 0 \\ I_2 & 0 & 0 \end{pmatrix}. \quad (26)$$

If  $N$  has one eigenvalue  $\lambda$ ,  $\dim S_0 = 2$ , and  $n = 6$ , then  $\{N, H\}$  is unitarily similar to  $\{(27), (28)\}$ :

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & ir_1 & 0 \\ 0 & \lambda & 0 & 1 & r_2 & ir_1 \\ 0 & 0 & \lambda & 0 & z & 0 \\ 0 & 0 & 0 & \lambda & 0 & z \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \quad |z| = 1, \quad z \neq -1, \\ r_1, r_2 \in \Re, \quad r_2 > 0, \quad (27)$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_2 & 0 \\ I_2 & 0 & 0 \end{pmatrix}. \quad (28)$$

If  $N$  has one eigenvalue  $\lambda$ ,  $\dim S_0 = 2$ , and  $n = 7$ , then  $\{N, H\}$  is unitarily similar to  $\{(29), (30)\}$ :

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & -z_1 \bar{z}_2 \cos \alpha & \sin \alpha \cos \beta \\ 0 & 0 & 0 & \lambda & 0 & z_1 \sin \alpha & z_2 \cos \alpha \cos \beta \\ 0 & 0 & 0 & 0 & \lambda & 0 & \sin \beta \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$|z_1| = |z_2| = 1, \quad 0 < \alpha, \beta \leq \pi/2, \\ z_1 = 1 \text{ if } \beta = \pi/2, \quad z_2 = 1 \text{ if } \alpha = \pi/2, \quad (29)$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_3 & 0 \\ I_2 & 0 & 0 \end{pmatrix}. \quad (30)$$

If  $N$  has one eigenvalue  $\lambda$ ,  $\dim S_0 = 2$ , and  $n = 8$ , then  $\{N, H\}$  is unitarily similar to  $\{(31), (32)\}$ :

$$N - \lambda I = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -z_1 \bar{z}_2 \sin \alpha \cos \beta & \cos \alpha \cos \gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & z_1 \cos \alpha \cos \beta & z_2 \sin \alpha \cos \gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin \gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$|z_1| = |z_2| = 1, 0 \leq \alpha < \pi/2, 0 < \beta < \gamma \leq \pi/2, \\ z_1 = 1 \text{ if } \gamma = \pi/2, z_2 = 1 \text{ if } \alpha = 0 \quad (31)$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_4 & 0 \\ I_2 & 0 & 0 \end{pmatrix}. \quad (32)$$

The following sections contain the proof of this theorem.

## 4 Two Distinct Eigenvalues of $N$

Suppose an indecomposable  $H$ -normal operator  $N$  has 2 distinct eigenvalues. Then (Lemma 1, [1])  $C^n = \mathcal{Q}_1 \dot{+} \mathcal{Q}_2$ ,  $\dim \mathcal{Q}_1 = \dim \mathcal{Q}_2 = m$ ,  $[\mathcal{Q}_1, \mathcal{Q}_1] = 0$ ,  $[\mathcal{Q}_2, \mathcal{Q}_2] = 0$ ,  $N\mathcal{Q}_1 \subseteq \mathcal{Q}_1$ ,  $N\mathcal{Q}_2 \subseteq \mathcal{Q}_2$ ,  $N_1 = N|_{\mathcal{Q}_1}$  ( $N_2 = N|_{\mathcal{Q}_2}$ ) has only one eigenvalue  $\lambda_1$  ( $\lambda_2$ ). According to Theorem 1,  $m = 2$  and  $n = 4$ . Note that the subspaces  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  are determined up to interchanging. Since  $N$  is indecomposable, at least one of the operators  $N_1, N_2$  is not scalar. Consequently, one can assume  $N_1 \neq \lambda_1 I$ . If both  $N_1$  and  $N_2$  are not scalar, then we can fix  $\mathcal{I}m\{\lambda_1 - \lambda_2\} > 0$  if  $\mathcal{I}m\{\lambda_1 - \lambda_2\} \neq 0$  and  $\mathcal{R}e\{\lambda_1 - \lambda_2\} > 0$  if  $\mathcal{I}m\{\lambda_1 - \lambda_2\} = 0$  (let us remember that  $\lambda_1 \neq \lambda_2$ ). Now  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  are determined uniquely.

As  $H$  is nondegenerate, for any basis in  $\mathcal{Q}_1$  there exists a basis in  $\mathcal{Q}_2$  such that

$$H = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Let us fix a basis in  $\mathcal{Q}_1$  such that

$$N_1 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}. \quad (33)$$

$N$  is  $H$ -normal if and only if

$$N_1 N_2^* = N_2^* N_1. \quad (34)$$

From (34) it follows that  $N_2^* = \alpha N_1 + \beta I$ . As  $N_2 = \bar{\alpha} N_1^* + \bar{\beta} I$  has the only eigenvalue  $\lambda_2$ , we conclude  $N_2 = \lambda_2 I + x(N_1^* - \bar{\lambda}_1 I)$  ( $x \in C$ ). Thus, we have reduced  $N$  to the form

$$N = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \oplus \begin{pmatrix} \lambda_2 & 0 \\ x & \lambda_2 \end{pmatrix}, x \in C. \quad (35)$$

Show that forms (35) with different values of  $x$  are not  $H$ -unitarily

$$NT = T\tilde{N}, \quad (36)$$

$$TT^{[*]} = I, \quad (37)$$

where  $N = N_1 \oplus N_2$ ,  $\tilde{N} = N_1 \oplus \tilde{N}_2$ ,  $N_1$  has form (33),

$$N_2 = \begin{pmatrix} \lambda_2 & 0 \\ x & \lambda_2 \end{pmatrix}, \tilde{N}_2 = \begin{pmatrix} \lambda_2 & 0 \\ \tilde{x} & \lambda_2 \end{pmatrix}.$$

From (36) it follows that  $T$  is block diagonal with respect to the decomposition  $C^n = \mathcal{Q}_1 \dot{+} \mathcal{Q}_2$ :  $T = T_1 \oplus T_2$ ,  $T_1$  satisfying the condition  $N_1 = T_1^{-1} N_1 T_1$ . Taking into account (37), we get  $T_2 = T_1^{*-1}$ , therefore,  $\tilde{N}_2 = T_2^{-1} N_2 T_2 = N_2$ , i.e.,  $\tilde{x} = x$ .

It can easily be checked that (35) is indecomposable so that we have proved the following lemma:

**Lemma 1** *If an indecomposable  $H$ -normal operator acts in a space  $C^n$  of rank 2 and has 2 distinct eigenvalues  $\lambda_1, \lambda_2$ , then  $n = 4$  and the pair  $\{N, H\}$  is unitarily similar to canonical pair  $\{(4), (5)\}$ :*

$$N = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & x & \lambda_2 \end{pmatrix}, x \in C,$$

for  $x \neq 0$   $\left[ \begin{array}{ll} \operatorname{Im}\{\lambda_1 - \lambda_2\} > 0 & \text{if } \operatorname{Im}\{\lambda_1 - \lambda_2\} \neq 0, \\ \operatorname{Re}\{\lambda_1 - \lambda_2\} > 0 & \text{otherwise,} \end{array} \right.$

$$H = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},$$

where the number  $x$  forms a complete and minimal invariant of the pair  $\{N, H\}$  under the unitary similarity (in short, we say that  $x$  is an  $H$ -unitary invariant). In other words, every pair  $\{N, H\}$  satisfying the hypothesis of the lemma is unitary similar to pair  $\{(4), (5)\}$  and pairs  $\{(4), (5)\}$  with different values of  $x$  are not  $H$ -unitarily similar to each other.

## 5 One Eigenvalue of $N$

Throughout what follows we will assume that  $N$  has only one eigenvalue  $\lambda$  so that  $N$  and  $H$  have form (2). Since the neutral subspace  $S_0$  cannot be more than two-dimensional, there appear two cases to be considered:  $\dim S_0 = 1$  and  $\dim S_0 = 2$ . Now let us prove the following proposition which holds for all spaces with indefinite scalar product:

**Proposition 2** *An  $H$ -normal operator such that  $\dim S_0 = 1$  is indecomposable.*

**Proof:** Assume the converse. Suppose some nondegenerate subspace  $V$  is invariant both for  $N$  and for  $N^{[*]}$ . Let us denote  $V_1 = V$ ,  $V_2 = V^{\perp}$ ,  $N_1 = N|_{V_1}$ ,  $N_2 = N|_{V_2}$ ,  $H_1 = H|_{V_1}$ ,  $H_2 = H|_{V_2}$ . The following conditions must hold:  $N_1 N_1^{[*]} = N_1^{[*]} N_1$ ,  $N_2 N_2^{[*]} = N_2^{[*]} N_2$ . Here  $N_i^{[*]}$  is the  $H_i$ -adjoint of  $N_i$  ( $i = 1, 2$ ). Let us define

$$S_0^i = \{x \in V_i : (N_i - \lambda I)x = (N_i^{[*]} - \bar{\lambda} I)x = 0\}, \quad i = 1, 2.$$

Since the operators  $N_1$  and  $N_1^{[*]}$  ( $N_2$  and  $N_2^{[*]}$ ) commute,  $\dim S_0^i \geq 1$  ( $i = 1, 2$ ), therefore,  $\dim\{S_0 = S_0^1 + S_0^2\} \geq 2$ . This contradicts the condition  $\dim S_0 = 1$ . Thus,  $N$  is indecomposable.

If  $\dim S_0 = 1$ , then rank of  $S$  is equal to 1, therefore, to classify the internal operator  $N_1$  we may apply Theorem 1 from [1]. Since the indecomposability (or decomposability) of  $N_1$  is a property which does not change under the unitary similarity of the pair  $\{N_1, H_1\}$ , we must consider both the case when  $N_1$  is indecomposable and that when  $N_1$  is decomposable.

### 5.1 $\dim S_0 = 1$ and $N_1$ is Indecomposable

If  $N_1$  is indecomposable, then, according to Theorem 1,  $2 \leq \dim S \leq 4$  (recall that rank of  $S$  is equal to 1). Therefore,  $4 \leq n \leq 6$ . Let us consider the alternatives  $n = 4, 5, 6$  one after another.

#### 5.1.1 $n = 4$

According to Theorem 1 of [1], one can assume that  $N_1$  and  $H_1$  are reduced to the form

$$N_1 = \begin{pmatrix} \lambda & z \\ 0 & \lambda \end{pmatrix}, \quad |z| = 1, \quad H_1 = D_2.$$

Hence

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & z & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H = D_4.$$

Throughout what follows only  $H$ -unitary transformations are used unless otherwise stipulated. This means that for each case we fix some form of the matrix  $H$  and find out to what form it is possible to reduce the matrix  $N$  without the change of  $H$ .

The condition of the  $H$ -normality of  $N$  is equivalent to the system

$$a\bar{z} = \bar{e}z \quad (38)$$

$$\mathcal{R}e\{a\bar{b}\} = \mathcal{R}e\{d\bar{e}\}. \quad (39)$$

If  $a = 0$ , then  $e = 0$ , therefore, the vector  $v_2$  from  $S$  ( $v_i$  are the basis vectors) belongs to  $S_0$ , which is impossible. Thus,  $a \neq 0$ . Replace the vector  $v_1$  by  $av_1$  and  $v_4$  by  $v_4/\bar{a}$ . This transformation reduces  $N - \lambda I$  to the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & b' & c' \\ 0 & 0 & z & d' \\ 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Further, apply the transformation

$$T = \begin{pmatrix} 1 & z\bar{d}' & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\bar{z}d' \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to the matrix  $N - \lambda I$ . We obtain:

$$N - \lambda I = \begin{pmatrix} 0 & 1 & b'' & c'' \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows from (39) that  $b'' = ir_1$  ( $r_1 \in \mathfrak{R}$ ). Taking the transformation

$$T = \begin{pmatrix} 1 & 0 & \frac{1}{2}z\bar{z}\mathcal{R}e\{c''\bar{z}\} & 0 \\ 0 & 1 & 0 & -\frac{1}{2}z\mathcal{R}e\{c''\bar{z}\} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we reduce  $N - \lambda I$  to the final form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & ir_1 & ir_2z \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad |z| = 1, \quad r_1, r_2 \in \mathfrak{R}, \quad (40)$$

where  $r_2 = \mathcal{I}m\{c''\bar{z}\}$ .

Let us prove that the numbers  $z$ ,  $r_1$ ,  $r_2$  are  $H$ -unitary invariants. Indeed, let  $T$  be an  $H$ -unitary transformation of the matrix  $N$  to the form  $\tilde{N}$ , where

$$\tilde{N} - \lambda I = \begin{pmatrix} 0 & 1 & i\tilde{r}_1 & i\tilde{r}_2\tilde{z} \\ 0 & 0 & \tilde{z} & 0 \\ 0 & 0 & 0 & \tilde{z}^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad |\tilde{z}| = 1, \quad \tilde{r}_1, \tilde{r}_2 \in \mathfrak{R}.$$

This means that  $T$  satisfies conditions (36) and (37). From Corollary of Proposition 1 it follows that  $T$  is block triangular with respect to the decomposition  $C^n = S_0 + S + S_1$ . According to Theorem 1 from [1],  $z$  is an  $H_1$ -unitary invariant of  $N_1$ .  $T_4 = T|_S$  is a  $H_1$ -unitary transformation of  $N_1$  to the form  $\tilde{N}_1$ , therefore,  $z$  is also an  $H$ -unitary invariant of  $N$ , i.e.,  $\tilde{z} = z$ . Applying condition (36), we see that  $T$  is uppertriangular and its diagonal terms are equal to each other. From (37) it follows that  $|t_{11}| = 1$ . Therefore, without loss of generality one can assume that  $t_{11} = 1$  (we replace our matrix  $T$  by the matrix  $T' = \bar{t}_{11}T$ ; the latter has the same properties (36), (37)).

Thus,

$$T = \begin{pmatrix} 1 & t_{12} & t_{13} & t_{14} \\ 0 & 1 & t_{23} & t_{24} \\ 0 & 0 & 1 & t_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For  $T$  to be  $H$ -unitary it is necessary and sufficient to have

$$\overline{t_{34}} + t_{12} = 0 \quad (41)$$

$$\overline{t_{24}} + \overline{t_{23}}t_{12} + t_{13} = 0 \quad (42)$$

$$\mathcal{R}et_{14} + \mathcal{R}e\{t_{12}\overline{t_{13}}\} = 0 \quad (43)$$

$$\mathcal{R}et_{23} = 0, \quad (44)$$

for  $T$  to reduce  $N$  to the form  $\tilde{N}$  it is necessary and sufficient to have

$$t_{23} + ir_1 = i\tilde{r}_1 + zt_{12} \quad (45)$$

$$t_{24} + ir_1t_{34} + ir_2z = i\tilde{r}_2z + z^2t_{13} \quad (46)$$

$$zt_{34} = z^2t_{23}. \quad (47)$$

Express  $t_{34}$  in terms of  $t_{23}$  from (47) and  $t_{12}$  in terms of  $t_{23}$  from (45):  $t_{34} = zt_{23}$ ,  $t_{12} = \overline{z}(ir_1 - i\tilde{r}_1) + \overline{z}t_{23}$ . Substituting these expressions in (41), we get:  $2\mathcal{R}et_{23} = i(\tilde{r}_1 - r_1)$ . Since  $\mathcal{R}et_{23} = 0$  (condition (44)),  $\tilde{r}_1 = r_1$ . Further, let us express  $t_{24}$  in terms of  $t_{13}$  and  $t_{23}$  (condition (46)):  $t_{24} = (i\tilde{r}_2 - ir_2)z + z^2t_{13} - ir_1zt_{23}$ . Then condition (42) can be written in the form

$$(ir_2 - i\tilde{r}_2) + \overline{zt_{13}} + zt_{13} + ir_1\overline{t_{23}} + |t_{23}|^2 = 0.$$

As  $\mathcal{R}et_{23} = 0$ ,  $ir_1\overline{t_{23}} \in \mathfrak{R}$ , consequently,  $\overline{zt_{13}} + zt_{13} + ir_1\overline{t_{23}} + |t_{23}|^2 \in \mathfrak{R}$ . But  $i(r_2 - \tilde{r}_2) \in \mathfrak{S}$ . Therefore,  $\tilde{r}_2 = r_2$ . Thus, the numbers  $z$ ,  $r_1$ ,  $r_2$  are  $H$ -unitary invariants.

Due to Proposition 2 matrix (40) is indecomposable so that we have proved the following lemma:

**Lemma 2** *If an indecomposable  $H$ -normal operator  $N$  ( $N : C^4 \rightarrow C^4$ ) has the only eigenvalue  $\lambda$ ,  $\dim S_0 = 1$ , the internal operator  $N_1$  is indecomposable, then the pair  $\{N, H\}$  is unitarily similar to canonical pair  $\{(6), (7)\}$ :*

$$N = \begin{pmatrix} \lambda & 1 & ir_1 & ir_2z \\ 0 & \lambda & z & 0 \\ 0 & 0 & \lambda & z^2 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad |z| = 1, \quad r_1, r_2 \in \mathfrak{R},$$

$$H = D_4,$$

where  $z$ ,  $r_1$ ,  $r_2$  are  $H$ -unitary invariants.

### 5.1.2 $n = 5$

According to Theorem 1 of [1], it can be assumed that the pair  $\{N_1, H_1\}$  has either form (48) or (49):

$$N_1 = \begin{pmatrix} \lambda & z & r \\ 0 & \lambda & z \\ 0 & 0 & \lambda \end{pmatrix}, \quad |z| = 1, \quad 0 < \arg z < \pi, \quad r \in \mathfrak{R}, \quad H_1 = D_3, \quad (48)$$

$$N_1 = \begin{pmatrix} \lambda & 1 & ir \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \quad r \in \mathfrak{R}, \quad H_1 = D_3. \quad (49)$$

For a while we will consider both the cases together, assuming that

$$N_1 = \begin{pmatrix} \lambda & z' & x \\ 0 & \lambda & z' \\ 0 & 0 & \lambda \end{pmatrix}, \quad |z'| = 1, \quad 0 \leq \arg z' < \pi, \quad x \in C.$$

Then

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c & d \\ 0 & 0 & z' & x & e \\ 0 & 0 & 0 & z' & f \\ 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The condition of the  $H$ -normality is equivalent to the system

$$a\bar{z}' = \bar{g}z' \quad (50)$$

$$a\bar{x} + b\bar{z}' = \bar{g}x + \bar{f}z' \quad (51)$$

$$2\mathcal{R}e\{a\bar{c}\} + |b|^2 = 2\mathcal{R}e\{e\bar{g}\} + |f|^2. \quad (52)$$

As above (see the case when  $n = 4$ ), one can check that  $a \neq 0$ , hence  $a$  can be assumed equal to 1, so  $g = z'^2$ . Having in mind these equalities, take the ( $H$ -unitary) transformation

$$T = \begin{pmatrix} 1 & \bar{z}'b & \bar{z}'(c - x\bar{z}'b) & 0 & -\frac{1}{2}|c - x\bar{z}'b|^2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -z'(\bar{c} - \bar{x}z'\bar{b}) \\ 0 & 0 & 0 & 1 & -z'\bar{b} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It reduces  $N - \lambda I$  to the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & d' \\ 0 & 0 & z' & x & e' \\ 0 & 0 & 0 & z' & f' \\ 0 & 0 & 0 & 0 & z'^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now apply either the transformation

$$T = \begin{pmatrix} 1 & 0 & 0 & \mathcal{R}ed' / (\mathcal{R}ez'^2 + 1) & 0 \\ 0 & 1 & 0 & 0 & -\mathcal{R}ed' / (\mathcal{R}ez'^2 + 1) \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (z' \neq i)$$

or

$$T = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2}i\mathcal{I}md' & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2}i\mathcal{I}md' \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (z' = i)$$

to the matrix  $N - \lambda I$ . We get

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & i(\mathcal{I}md' + \mathcal{I}m\{d'\bar{z}'^2\}) / (\mathcal{R}ez'^2 + 1) \\ 0 & 0 & z' & x & e' \\ 0 & 0 & 0 & z' & f' \\ 0 & 0 & 0 & 0 & z'^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (z' \neq i)$$

or

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & \mathcal{R}ed' \\ 0 & 0 & i & x & e' \\ 0 & 0 & 0 & i & f' \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (z' = i).$$

Now we shall distinguish cases (48) and (49).

(a)  $z' = 1$ ,  $x = ir_1$  ( $r_1 \in \mathfrak{R}$ ). Conditions (51), (52) of the  $H$ -normality of  $N$  yield:  $f' = 2ir_1$ ,  $e' = -2r_1^2 + ir_2$ . Denote  $(\mathcal{I}md' - \mathcal{I}m\{d'z'^2\})/(\mathcal{R}ez'^2 + 1)$  by  $r_3$ . We have

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & ir_3 \\ 0 & 0 & 1 & ir_1 & -2r_1^2 + ir_2 \\ 0 & 0 & 0 & 1 & 2ir_1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad r_1, r_2, r_3 \in \mathfrak{R}. \quad (53)$$

There remains to check the  $H$ -unitary invariance of the numbers  $r_1$ ,  $r_2$ ,  $r_3$ . To prove this, let us suppose that some  $H$ -unitary matrix  $T$  reduces (53) to the form

$$\tilde{N} - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & i\tilde{r}_3 \\ 0 & 0 & 1 & i\tilde{r}_1 & -2\tilde{r}_1^2 + i\tilde{r}_2 \\ 0 & 0 & 0 & 1 & 2i\tilde{r}_1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{r}_1, \tilde{r}_2, \tilde{r}_3 \in \mathfrak{R}.$$

From condition (36)  $NT = T\tilde{N}$  it follows that  $T$  is uppertriangular with diagonal terms which are equal to each other. According to Theorem 1 from [1],  $r_1$  is an  $H_1$ -unitary invariant for  $N_1$ . We already know that in this case  $r_1$  must be an  $H$ -unitary invariant (see the previous case  $n = 4$ ), i.e.,  $\tilde{r}_1 = r_1$ . For  $T$  to be  $H$ -unitary, i.e., to satisfy (37),  $|t_{11}|$  must be equal to 1. Therefore, as in case  $n = 4$ , one can assume that  $t_{11} = 1$ . Thus,  $T$  has the form

$$T = \begin{pmatrix} 1 & t_{12} & t_{13} & t_{14} & t_{15} \\ 0 & 1 & t_{23} & t_{24} & t_{25} \\ 0 & 0 & 1 & t_{34} & t_{35} \\ 0 & 0 & 0 & 1 & t_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (54)$$

Condition (36) amounts to system (55) - (60), (37) to system (61) - (66):

$$t_{23} = t_{12} \quad (55)$$

$$t_{24} = ir_1 t_{12} + t_{13} \quad (56)$$

$$t_{25} + ir_3 = i\tilde{r}_3 + (-2r_1^2 + i\tilde{r}_2)t_{12} + 2ir_1 t_{13} + t_{14} \quad (57)$$

$$t_{34} = t_{23} \quad (58)$$

$$t_{35} + ir_1 t_{45} + ir_2 = i\tilde{r}_2 + 2ir_1 t_{23} + t_{24} \quad (59)$$

$$t_{45} = t_{34}, \quad (60)$$

$$\overline{t_{45}} + t_{12} = 0 \quad (61)$$

$$\overline{t_{35}} + \overline{t_{34}}t_{12} + t_{13} = 0 \quad (62)$$

$$\overline{t_{25}} + \overline{t_{24}}t_{12} + \overline{t_{23}}t_{13} + t_{14} = 0 \quad (63)$$

$$2\mathcal{R}et_{15} + 2\mathcal{R}e\{t_{12}\overline{t_{14}}\} + |t_{13}|^2 = 0 \quad (64)$$

$$\overline{t_{34}} + t_{23} = 0 \quad (65)$$

$$2\mathcal{R}et_{24} + |t_{23}|^2 = 0. \quad (66)$$

Express  $t_{35}$  in terms of  $t_{23}$ ,  $t_{24}$ ,  $t_{45}$  from (59) and substitute this expression in (62), taking into account that  $t_{12} = t_{23} = t_{34} = t_{45}$  and expressing  $t_{24}$  in terms of  $t_{12}$  and  $t_{13}$  from condition (56). We obtain  $ir_2 - i\tilde{r}_2 = 2ir_1\overline{t_{12}} + 2\mathcal{R}et_{13} + |t_{12}|^2$ . Since  $\mathcal{R}et_{12} = 0$  (equation (61)), we have  $2ir_1\overline{t_{12}} \in \mathfrak{R}$ , hence, the right hand side of the condition obtained is real and the left one is imaginary. Therefore,  $\tilde{r}_2 = r_2$ .

Since  $t_{13} = t_{24} - ir_1 t_{12}$  (condition (56)),  $t_{25}$  can be expressed in terms of  $t_{12}$ ,  $t_{24}$  and  $t_{14}$  in the following way (see condition (57)):  $t_{25} = i(\tilde{r}_3 - r_3) + ir_2 t_{12} + 2ir_1 t_{24} + t_{14}$ . By substituting this expression in (63), we get  $ir_3 - i\tilde{r}_3 = ir_2\overline{t_{12}} + ir_1(2\overline{t_{24}} + |t_{12}|^2) + 2\mathcal{R}e\{t_{12}\overline{t_{24}}\} + 2\mathcal{R}et_{14}$ . Because of condition (66)  $ir_1(2\overline{t_{24}} + |t_{12}|^2)$  is

real as well as the rest terms of the right hand side, hence,  $\tilde{r}_3 = r_3$ . We have proved the  $H$ -unitary invariance of  $r_1, r_2, r_3$ .

(b)  $z' = z, |z| = 1, 0 < \arg z < \pi, x = r_1 \in \mathfrak{R}$ . Applying conditions (51), (52) of the  $H$ -normality of  $N$ , we get

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & ir_3 \\ 0 & 0 & z & r_1 & -2z^2r_1^2Im^2z + ir_2z^2 \\ 0 & 0 & 0 & z & -2ir_1z^2Imz \\ 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad r_1, r_2, r_3 \in \mathfrak{R} \quad (z \neq i)$$

or

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & r_3 \\ 0 & 0 & i & r_1 & 2r_1^2 + ir_2 \\ 0 & 0 & 0 & i & 2ir_1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad r_1, r_2, r_3 \in \mathfrak{R} \quad (z = i).$$

We shall join these cases, assuming that

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & ix \\ 0 & 0 & z & r_1 & -2z^2r_1^2Im^2z + ir_2z^2 \\ 0 & 0 & 0 & z & -2ir_1z^2Imz \\ 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$x = \begin{cases} r_3 \in \mathfrak{R}, & z \neq i \\ -ir_3 \in \mathfrak{S} (r_3 \in \mathfrak{R}), & z = i. \end{cases}$$

Let us prove the  $H$ -unitary invariance of the numbers  $z, r_1, r_2, r_3$  (or  $x$ ). Suppose some matrix  $T$  realizes the  $H$ -unitary transformation of  $N$  to the form  $\tilde{N}$ , where

$$\tilde{N} - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & i\tilde{x} \\ 0 & 0 & \tilde{z} & \tilde{r}_1 & -2\tilde{z}^2\tilde{r}_1^2Im^2\tilde{z} + i\tilde{r}_2\tilde{z}^2 \\ 0 & 0 & 0 & \tilde{z} & -2i\tilde{r}_1\tilde{z}^2Im\tilde{z} \\ 0 & 0 & 0 & 0 & \tilde{z}^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By Theorem 1 of [1],  $z$  and  $r_1$  are  $H_1$ -unitary invariants, hence, they are  $H$ -unitary invariants, i.e.,  $\tilde{z} = z, \tilde{r}_1 = r_1$ . Further, from (36) it follows that  $T$  is uppertriangular with diagonal terms which are equal to each other. Applying (37), we get that  $T$  has form (54). Now condition (37) is equivalent to system (61) - (66), condition (36) to system (67) - (72):

$$t_{23} = zt_{12} \quad (67)$$

$$t_{24} = r_1t_{12} + zt_{13} \quad (68)$$

$$t_{25} + ix = i\tilde{x} + (-2z^2r_1^2Im^2z + i\tilde{r}_2z^2)t_{12} - 2ir_1z^2Imz t_{13} + z^2t_{14} \quad (69)$$

$$t_{34} = t_{23} \quad (70)$$

$$zt_{35} + r_1t_{45} + ir_2z^2 = i\tilde{r}_2z^2 - 2ir_1z^2Imz t_{23} + z^2t_{24} \quad (71)$$

$$zt_{45} = z^2t_{34}. \quad (72)$$

Express  $t_{35}$  in terms of  $t_{23}, t_{24}, t_{45}$  and, taking into account the equalities  $t_{12} = \bar{z}t_{23}$  (67),  $t_{13} = \bar{z}(t_{24} - r_1t_{12})$  (68),  $t_{34} = t_{23}$  (70),  $t_{45} = zt_{23}$  (72), substitute the obtained expression in (62). After multiplying both sides by  $\bar{z}$ , we have:  $(ir_2 - i\tilde{r}_2) = -2ir_1Imz t_{23} + t_{24} + \bar{t}_{24} + |t_{23}|^2 - r_1(\bar{z}t_{23} + z\bar{t}_{23})$ . Since  $\mathcal{R}et_{23} = 0$  (65), the right hand side of this equality is real. Consequently,  $\tilde{r}_2 = r_2$ .

Now let us express  $t_{25}$  in terms of  $t_{23}, t_{24}, t_{14}$  from (69):  $t_{25} = i(\tilde{x} - x) - 2r_1^2zIm^2z t_{23} + ir_2zt_{23} - 2ir_1zImz t_{24} + 2ir_1^2Imz t_{23} + z^2t_{14}$ . Rewrite condition (63) in the form  $t_{25} + t_{24}\bar{t}_{12} + t_{23}\bar{t}_{13} + \bar{t}_{14} = 0$ , multiply

its both sides by  $\bar{z}$  and substitute the expression for  $t_{25}$  in it. We obtain:  $i(x - \tilde{x})\bar{z} = -2r_1^2 Im^2 z t_{23} + ir_2 t_{23} - 2ir_1 Imz t_{24} + 2ir_1^2 \bar{z} Imz t_{23} + z t_{14} + \bar{z} \bar{t}_{14} + t_{23} \bar{t}_{24} + t_{24} \bar{t}_{23} - z r_1 |t_{23}|^2$ . Since  $-2r_1^2 Im^2 z + 2ir_1^2 \bar{z} Imz = ir_1^2 Imz Re z$  and  $-2ir_1 Imz t_{24} - r_1 z |t_{23}|^2 = r_1 (2Re z Re t_{24} + 2Imz Im t_{24})$ , the right hand side is real. Therefore,  $Im[i\bar{z}(x - \tilde{x})] = 0$ . If  $z \neq i$ , then this condition means  $(r_3 - \tilde{r}_3)Re z = 0$ , hence  $\tilde{r}_3 = r_3$  because  $Re z \neq 0$ . If  $z = i$ , then  $Im[i(\tilde{r}_3 - r_3)] = 0$ , hence also we get  $\tilde{r}_3 = r_3$ . This concludes the proof of the  $H$ -unitary invariance of  $z, r_1, r_2, r_3$ .

Due to Proposition 2 all obtained forms are indecomposable. They are not  $H$ -unitarily similar because their internal matrices  $N_1$  are not  $H_1$ -unitarily similar due to Theorem 1 of [1]. Thus, we have proved the following lemma:

**Lemma 3** *If an indecomposable  $H$ -normal operator  $N$  ( $N : C^5 \rightarrow C^5$ ) has the only eigenvalue  $\lambda$ ,  $\dim S_0 = 1$ , the internal operator  $N_1$  is indecomposable, then the pair  $\{N, H\}$  is unitarily similar to one and only one of canonical pairs  $\{(8), (11)\}, \{(9), (11)\}, \{(10), (11)\}$ :*

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 & ir_3 \\ 0 & \lambda & 1 & ir_1 & -2r_1^2 + ir_2 \\ 0 & 0 & \lambda & 1 & 2ir_1 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \quad r_1, r_2, r_3 \in \mathfrak{R},$$

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 & ir_3 \\ 0 & \lambda & z & r_1 & -2z^2 r_1^2 Im^2 z + ir_2 z^2 \\ 0 & 0 & \lambda & z & -2ir_1 z^2 Imz \\ 0 & 0 & 0 & \lambda & z^2 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \begin{array}{l} |z| = 1, z \neq i, \\ 0 < \arg z < \pi, \\ r_1, r_2, r_3 \in \mathfrak{R}, \end{array}$$

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 & r_3 \\ 0 & \lambda & i & r_1 & 2r_1^2 + ir_2 \\ 0 & 0 & \lambda & i & 2ir_1 \\ 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \quad r_1, r_2, r_3 \in \mathfrak{R},$$

$$H = D_5,$$

where  $z, r_1, r_2, r_3$  are  $H$ -unitary invariants.

### 5.1.3 $n = 6$

In this case, according to Theorem 1 from [1], the matrices  $N_1$  and  $H_1$  can be written in the form

$$N_1 = \begin{pmatrix} \lambda & \cos \alpha & \sin \alpha & 0 \\ 0 & \lambda & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad 0 < \alpha \leq \pi/2, \quad H_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so that

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c & d & e \\ 0 & 0 & \cos \alpha & \sin \alpha & 0 & f \\ 0 & 0 & 0 & 0 & 1 & g \\ 0 & 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The condition of the  $H$ -normality of  $N$  is equivalent to the following system:

$$a = \bar{p} \cos \alpha \tag{73}$$

$$0 = \bar{p} \sin \alpha \tag{74}$$

$$\begin{aligned} b \cos \alpha + c \sin \alpha &= \bar{g} \\ 2Re\{\bar{a}d\} + |b|^2 + |c|^2 &= 2Re\{f\bar{p}\} + |g|^2 + |h|^2. \end{aligned}$$

From (74) and the condition  $0 < \alpha \leq \pi/2$  it follows that  $p = 0$ . Then from (73) we obtain that also  $a = 0$ . Hence, the vector  $v_2 \in S$  belongs to  $S_0$ , which is impossible. This contradiction proves that for indecomposable operator  $N : C^6 \rightarrow C^6$   $\dim S_0 \neq 1$ .

Recall that if  $n > 6$ , then the operator  $N_1$  is always decomposable (Theorem 1 of [1]). Thus, we have obtained the classification for all indecomposable operators  $N$  having also indecomposable internal operator  $N_1$ .

## 5.2 $\dim S_0 = 1$ and $N_1$ is Decomposable

If the operator  $N_1$  is decomposable, then it can be represented as an orthogonal sum of indecomposable operators  $N_1^{(1)}, \dots, N_1^{(p)} : N_1 = N_1^{(1)} \oplus \dots \oplus N_1^{(p)}$ ,  $H_1 = H_1^{(1)} \oplus \dots \oplus H_1^{(p)}$ . Without loss of generality it can be assumed that  $H_1^{(1)}$  has one negative eigenvalue. Denote  $H_1^{(1)}$  by  $H_2$ ,  $N_1^{(1)}$  by  $N_2$ ,  $H_1^{(2)} \oplus \dots \oplus H_1^{(p)}$  by  $H_3$ ,  $N_1^{(2)} \oplus \dots \oplus N_1^{(p)}$  by  $N_3$ . Since  $H_3$  has only positive eigenvalues, one can assume that  $H_3 = I$ .  $N_3$  is a usual normal operator having the only eigenvalue  $\lambda$ , hence,  $N_3 = \lambda I$ .

Show that the size of  $N_3$  is equal to  $1 \times 1$ . Indeed, let  $\dim V_2 = k$ ,  $\dim V_3 = l > 1$  ( $V_2$  and  $V_3$  are the subspaces of  $S$  corresponding to  $N_2$  and  $N_3$ , respectively),  $V_2 = \text{span}\{w_1^{(2)}, w_2^{(2)}, \dots, w_k^{(2)}\}$ ,  $V_3 = \text{span}\{w_1^{(3)}, w_2^{(3)}, \dots, w_l^{(3)}\}$ . Then, by the above,

$$N = \begin{pmatrix} \lambda & M_1 & M_2 & * \\ 0 & N_2 & 0 & * \\ 0 & 0 & \lambda I & * \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad N^{[*]} = \begin{pmatrix} \bar{\lambda} & M_3 & M_4 & * \\ 0 & N_2^{[*]} & 0 & * \\ 0 & 0 & \bar{\lambda} I & * \\ 0 & 0 & 0 & \bar{\lambda} \end{pmatrix},$$

where  $M_1 = (a_1, a_2, \dots, a_k)$ ,  $M_2 = (b_1, b_2, \dots, b_l)$ ,  $M_3 = (c_1, c_2, \dots, c_k)$ ,  $M_4 = (d_1, d_2, \dots, d_l)$ . Because of the  $H_2$ -normality of  $N_2$   $\dim S_0^{(2)} \geq 1$  ( $S_0^{(2)} = \{x \in V_2 : (N_2 - \lambda I)x = (N_2^{[*]} - \bar{\lambda} I)x = 0\}$ ), hence, without loss of generality it can be assumed that  $w_1^{(2)} \in S_0^{(2)}$ . Since  $l > 1$ ,  $\exists \{\alpha_i\}_1^{n+1}$  ( $\sum_1^{n+1} |\alpha_i| \neq 0$ ):

$$\sum_1^n \alpha_i b_i + \alpha_{n+1} a_1 = 0 \quad (75)$$

$$\sum_1^n \alpha_i d_i + \alpha_{n+1} c_1 = 0. \quad (76)$$

Therefore,  $\exists v = \sum_1^n \alpha_i w_i^{(3)} + \alpha_{n+1} w_1^{(2)} \neq 0 : (N - \lambda I)v = (N^{[*]} - \bar{\lambda} I)v = 0$ , i.e., some nonzero vector from  $S$  belongs to  $S_0$ . This is impossible so that  $\dim V_3 = 1$ .

As  $N_2$  is indecomposable and rank of  $V_2$  is less than or equal to 1,  $\dim V_2 \leq 4$  in accordance with Theorem 1. Thus,  $1 \leq \dim V_2 \leq 4$ ,  $\dim V_3 = 1$  so that  $4 \leq n \leq 7$ . Consider the cases  $n = 4, 5, 6, 7$  one after another.

### 5.2.1 $n = 4$

Then  $\dim V_2 = 1$ ,  $\dim V_3 = 1$ ,

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since  $H_1 = -1 \oplus 1$  is congruent to  $D_2$ , we will assume that  $H_1 = D_2$  so that  $H = D_4$ . Having fixed  $H = D_4$ , we will apply, as is customary, only  $H$ -unitary transformations.

The condition of the  $H$ -normality of  $N$  is now equivalent to the following:

$$\text{Re}\{\bar{a}b\} = \text{Re}\{d\bar{e}\}. \quad (77)$$

Since the assumption  $a = b = 0$  contradicts the condition  $S \cap S_0 = \{0\}$  (because then either  $v_2$  or  $v_3$  belongs to  $S_0$ ), one can assume that  $a \neq 0$  and, therefore,  $a = 1$  (see the paragraph after (39)). Keeping in mind that  $a = 1$ , reduce  $N - \lambda I$  to the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & b' = \operatorname{sgn} \mathcal{R}e b & c' \\ 0 & 0 & 0 & d' \\ 0 & 0 & 0 & e' \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

having applied either the transformation

$$T = \begin{pmatrix} \sqrt{|\mathcal{R}e b|} & 0 & 0 & 0 \\ 0 & \sqrt{|\mathcal{R}e b|} & -i \operatorname{Im} b / \sqrt{|\mathcal{R}e b|} & 0 \\ 0 & 0 & 1 / \sqrt{|\mathcal{R}e b|} & 0 \\ 0 & 0 & 0 & 1 / \sqrt{|\mathcal{R}e b|} \end{pmatrix} \quad (\mathcal{R}e b \neq 0)$$

or

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\mathcal{R}e b = 0).$$

Now consider the three cases ( $\mathcal{R}e b' = 0, 1$  or  $-1$ ) separately.

(a)  $b' = 0$ . Since  $\mathcal{R}e\{d'e'\} = 0$  (condition (77) of the  $H$ -normality of  $N$ ) and  $d' \neq 0$  (otherwise  $v_3 \in S_0$ ), the representation  $d' = \varrho_1 z$ ,  $e' = i \varrho_2 z$  ( $|z| = 1$ ,  $\varrho_1, \varrho_2 \in \mathfrak{R}$ ,  $\varrho_1 > 0$ ) is valid. Therefore, taking

$$T = \begin{pmatrix} \sqrt{\varrho_1} & 0 & 0 & 0 \\ 0 & \sqrt{\varrho_1} & 0 & 0 \\ 0 & i \varrho_2 / \sqrt{\varrho_1} & 1 / \sqrt{\varrho_1} & 0 \\ 0 & 0 & 0 & 1 / \sqrt{\varrho_1} \end{pmatrix},$$

we reduce  $N - \lambda I$  to the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & c'' \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

One can assume that  $c'' = 0$ . To achieve this it is sufficient to apply the transformation

$$T = \begin{pmatrix} 1 & 0 & \overline{c''} & 0 \\ 0 & 1 & 0 & -c'' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

There remains to prove that  $z$  is an  $H$ -unitary invariant. Indeed, any matrix  $T$  satisfying condition (36) ( $(N - \lambda I)T = T(\tilde{N} - \lambda I)$ ) for the matrices

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{N} - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \tilde{z} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad |z| = |\tilde{z}| = 1$$

and condition (37)  $TT^{[*]} = I$  has the form

$$T = t_{11} \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad |t_{11}| = 1.$$

This follows the desired equality  $z = \tilde{z}$ .

(b)  $b' = 1$ . As  $\mathcal{R}e\{d'e'\} = 1$  (condition (77)),  $d' = \varrho z$ ,  $e' = (1/\varrho + ir)z$  ( $|z| = 1$ ,  $\varrho, r \in \mathfrak{R}$ ,  $\varrho > 0$ ). Consider the transformation

$$T = I_1 \oplus \begin{pmatrix} -it/(1-it) & 1/(1-it) \\ 1/(1-it) & -it/(1-it) \end{pmatrix} \oplus I_1, \quad t \in \mathfrak{R}, \quad (78)$$

where  $t$  is a root of the equation  $1 + t^2 = 1/\varrho^2 + (t\varrho + r)^2$ . Its discriminant  $\mathcal{D}/4 = 1/\varrho^2 + \varrho^2 + r^2 - 2$  is nonnegative so that  $t$  is in fact real. Subjecting to (78), the matrix  $N - \lambda I$  becomes the following:

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 1 & c'' \\ 0 & 0 & 0 & z' \\ 0 & 0 & 0 & (1 + ir')z' \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad |z'| = 1, \quad r' \in \mathfrak{R}.$$

Note that if  $r' = 0$ , then there exists a nonzero vector  $v = \alpha v_2 + \beta v_3 \in S_0$ , which is impossible. Applying (78) with  $t = -\frac{1}{2}r'$ , we can replace  $r'$  by  $-r'$ . Thus, we can assume  $r' > 0$ . Finally, to get  $c'' = 0$  it is sufficient to take

$$T = \begin{pmatrix} 1 & t_{12} & t_{13} & -\mathcal{R}e\{t_{12}\overline{t_{13}}\} \\ 0 & 1 & 0 & -\overline{t_{13}} \\ 0 & 0 & 1 & -\overline{t_{12}} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where  $t_{12} = e^{-i\varphi/2}(rc_1'' - 2c_2'')/(2r)$ ,  $t_{13} = e^{-i\varphi/2}c_2''/r$  (we mean that  $z' = e^{i\varphi}$ ,  $c_1'' = \mathcal{R}e\{e''e^{-i\varphi/2}\}$ ,  $c_2'' = \mathcal{I}m\{e''e^{-i\varphi/2}\}$ ).

Thus, we have reduced the matrix  $N - \lambda I$  to the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & (1 + ir)z \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad |z| = 1, \quad r \in \mathfrak{R} > 0.$$

Now there remains to show that the numbers  $z$  and  $r$  are  $H$ -unitary invariants.

First note that for a block triangular matrix

$$T = \begin{pmatrix} T_1 & T_2 & T_3 \\ 0 & T_4 & T_5 \\ 0 & 0 & T_6 \end{pmatrix} \quad (79)$$

to reduce  $N - \lambda I$  to the form  $\tilde{N} - \lambda I$ , where

$$N - \lambda I = \begin{pmatrix} 0 & N_1 & N_2 \\ 0 & N_3 & N_4 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{N} - \lambda I = \begin{pmatrix} 0 & \widetilde{N}_1 & \widetilde{N}_2 \\ 0 & \widetilde{N}_3 & \widetilde{N}_4 \\ 0 & 0 & 0 \end{pmatrix},$$

it is necessary and sufficient to have

$$N_1 T_4 = T_1 \widetilde{N}_1 + T_2 \widetilde{N}_3 \quad (80)$$

$$N_1 T_5 + N_2 T_6 = T_1 \widetilde{N}_2 + T_2 \widetilde{N}_4 \quad (81)$$

$$N_3 T_4 = T_4 \widetilde{N}_3 \quad (82)$$

$$N_3 T_5 + N_4 T_6 = T_4 \widetilde{N}_4. \quad (83)$$

If

$$H = \begin{pmatrix} 0 & 0 & I \\ 0 & H_1 & 0 \\ I & 0 & 0 \end{pmatrix},$$

then for (79) to be  $H$ -unitary it is necessary and sufficient to have

$$T_1 T_6^* = I \quad (84)$$

$$T_4 H_1 T_2^* + T_5 T_1^* = 0 \quad (85)$$

$$T_1 T_3^* + T_2 H_1 T_2^* + T_3 T_1^* = 0 \quad (86)$$

$$T_4 H_1 T_4^* H_1 = I. \quad (87)$$

Since any  $H$ -unitary transformation  $T$  such that

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & (1+ir)z \\ 0 & 0 & 0 & 0 \end{pmatrix} T = T \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \tilde{z} \\ 0 & 0 & 0 & (1+i\tilde{r})\tilde{z} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$|z| = |\tilde{z}| = 1$ ,  $r, \tilde{r} \in \Re > 0$ , has to be block triangular (by Corollary of Proposition 1), systems (80) - (83), (84) - (87) are applicable. Combining (80) and (87), we get  $|t_{11}| = 1$ , hence (condition (84))  $t_{44} = t_{11}$ . Now from (80) and (83) it follows that  $(2+ir)z = (2+i\tilde{r})\tilde{z}$ , hence  $\tilde{z} = z$ ,  $\tilde{r} = r$ , Q.E.D.

(c)  $b' = -1$ . The matrix  $N - \lambda I$  can be carried into the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & -(1+ir)z \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad |z| = 1, \quad r \in \Re > 0,$$

where  $z$  and  $r$  are  $H$ -unitary invariants. The proof is analogous to the case (b) above.

Thus, we have obtained the canonical form for each case considered. By using conditions (80) - (87) one can easily check that these forms are not  $H$ -unitarily similar to each other. They are indecomposable due to Proposition 2. Thus, we have proved the following lemma:

**Lemma 4** *If an indecomposable  $H$ -normal operator  $N$  ( $N : C^4 \rightarrow C^4$ ) has the only eigenvalue  $\lambda$ ,  $\dim S_0 = 1$ , the internal operator  $N_1$  is decomposable, then the pair  $\{N, H\}$  is unitarily similar to one and only one of canonical pairs  $\{(12), (15)\}$ ,  $\{(13), (15)\}$ ,  $\{(14), (15)\}$ :*

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & z \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad |z| = 1,$$

$$N = \begin{pmatrix} \lambda & 1 & 1 & 0 \\ 0 & \lambda & 0 & z \\ 0 & 0 & \lambda & (1+ir)z \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad |z| = 1, \quad r \in \Re > 0,$$

$$N = \begin{pmatrix} \lambda & 1 & -1 & 0 \\ 0 & \lambda & 0 & z \\ 0 & 0 & \lambda & -(1+ir)z \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad |z| = 1, \quad r \in \Re > 0,$$

$$H = D_4,$$

where  $z, r$  are  $H$ -unitary invariants.

### 5.2.2 $n = 5$

Then  $\dim V_2 = 2$ ,  $\dim V_3 = 1$  and, according to Theorem 1 from [1], after interchanging the 3-rd and 4-th rows and colomns, we get:

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c & d \\ 0 & 0 & 0 & z & e \\ 0 & 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, |z| = 1, H = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The condition of the  $H$ -normality of  $N$  is equivalent to the system

$$a\bar{z} = \bar{g}z \quad (88)$$

$$2\Re\{a\bar{c}\} + |b|^2 = 2\Re\{e\bar{g}\} + |f|^2. \quad (89)$$

It is readily seen that  $a \neq 0$ , consequently, it can be assumed that  $a = 1$  and  $g = z^2$  (see the paragraph after (39)). Further, take the ( $H$ -unitary) transformation

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -b & -\frac{1}{2}|b|^2 & 0 \\ 0 & 0 & 1 & \frac{1}{b} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and reduce  $N - \lambda I$  to the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & c' & d' \\ 0 & 0 & 0 & z & e' \\ 0 & 0 & 0 & 0 & f' \\ 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Applying now the transformation

$$T = I_1 \oplus \begin{pmatrix} 1 & 0 & i\Im\{e'z^2\} \\ 0 & e^{i \arg f'} & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_1,$$

we get

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & c'' & d'' \\ 0 & 0 & 0 & z & r_1 z^2 \\ 0 & 0 & 0 & 0 & r_2 \\ 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, r_1, r_2 \in \Re, r_2 \geq 0.$$

We can assume that  $r_2 > 0$  because otherwise  $v_3 \in S_0$ , which is impossible. From condition (89) of the  $H$ -normality of  $N$  it follows that  $c'' = r_1 + \frac{1}{2}r_2^2 + ir_3$  ( $r_3 \in \Re$ ). Keeping in mind these conditions, apply the transformation

$$T = \begin{pmatrix} 1 & t_{12} & t_{13} & 0 & -\frac{1}{2}|t_{13}|^2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{t_{13}}{r_2} \\ 0 & 0 & 0 & 1 & -\frac{t_{12}}{r_2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where  $t_{12} = r_1\bar{z}$ ,  $t_{13} = (d'' - r_1z(r_1 + \frac{1}{2}r_2^2 + ir_3))/r_2$ , to the matrix  $N - \lambda I$ . Then  $c''' = \frac{1}{2}r_2^2 + ir_3$ ,  $d''' = 0$ , the rest terms of  $N - \lambda I$  do not change. Renaming  $r_2$  and  $r_3$ , write out the final form of  $N - \lambda I$ :

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 0 & \frac{1}{2}r_1^2 + ir_2 & 0 \\ 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & r_1 \\ 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, r_1, r_2 \in \Re, r_1 > 0, |z| = 1.$$

To prove the  $H$ -unitary invariance of  $z, r_1, r_2$  assume that

$$\tilde{N} - \lambda I = \begin{pmatrix} 0 & 1 & 0 & \frac{1}{2}\tilde{r}_1^2 + i\tilde{r}_2 & 0 \\ 0 & 0 & 0 & \tilde{z} & 0 \\ 0 & 0 & 0 & 0 & \tilde{r}_1 \\ 0 & 0 & 0 & 0 & \tilde{z}^2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tilde{r}_1, \tilde{r}_2 \in \mathfrak{R}, \tilde{r}_1 > 0, |\tilde{z}| = 1,$$

and there exists a matrix  $T$  such that  $NT = T\tilde{N}$  (condition (36)) and  $TT^{[*]} = I$  (condition (37)). Recall that  $T$  has block form (79) so that conditions (80) - (87) hold. From (82) it follows that  $t_{23} = 0$  and  $zt_{44} = \tilde{z}t_{22}$ . Since  $t_{22}\overline{t_{44}} = 1$  (87),  $z|t_{44}|^2 = \tilde{z}$ , i.e.,  $\tilde{z} = z$ ,  $|t_{44}| = 1$ . Therefore, one can assume that

$$T_4 = \begin{pmatrix} 1 & 0 & it \\ 0 & t_{33} & 0 \\ 0 & 0 & 1 \end{pmatrix}, |t_{33}| = 1, t \in \mathfrak{R}$$

because it is allowed to divide  $T$  by its term  $t_{22} = t_{44}$  of modulus 1. Now from (83) it follows that  $t_{45} = itz$ ,  $\tilde{r}_1 t_{33} = r_1$ . As  $r_1, \tilde{r}_1 > 0$ ,  $t_{33} = 1$  and  $\tilde{r}_1 = r_1$ . Since  $t_{12} = -\overline{t_{45}}$  (condition (85)) and  $t_{24} + (\frac{1}{2}r_1^2 + ir_2)t_{44} = (\frac{1}{2}\tilde{r}_1^2 + i\tilde{r}_2)t_{11} + \tilde{z}t_{12}$  (condition (80)),  $\tilde{r}_2 = r_2$ . This completes the proof of the  $H$ -unitary invariance of  $z, r_1, r_2$ .

Due to Proposition 2 the obtained form is indecomposable. Thus, we have proved the following lemma:

**Lemma 5** *If an indecomposable  $H$ -normal operator  $N$  ( $N : C^5 \rightarrow C^5$ ) has the only eigenvalue  $\lambda$ ,  $\dim S_0 = 1$ , the internal operator  $N_1$  is decomposable, then the pair  $\{N, H\}$  is unitarily similar to canonical pair  $\{(16), (17)\}$ :*

$$N = \begin{pmatrix} \lambda & 1 & 0 & \frac{1}{2}r_1^2 + ir_2 & 0 \\ 0 & \lambda & 0 & z & 0 \\ 0 & 0 & \lambda & 0 & r_1 \\ 0 & 0 & 0 & \lambda & z^2 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, |z| = 1, r_1, r_2 \in \mathfrak{R}, r_1 > 0,$$

$$H = D_5,$$

where  $r_1, r_2, z$  are  $H$ -unitary invariants.

### 5.2.3 $n = 6$

In this case  $\dim V_2 = 3, \dim V_3 = 1$ . The matrices  $N - \lambda I$  and  $H$ , according to Theorem 1 from [1], have the form:

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c & d & e \\ 0 & 0 & z & r & 0 & f \\ 0 & 0 & 0 & z & 0 & g \\ 0 & 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, |z| = 1, r \in \mathfrak{R} \quad (90)$$

or

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c & d & e \\ 0 & 0 & 1 & ir & 0 & f \\ 0 & 0 & 0 & 1 & 0 & g \\ 0 & 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, r \in \mathfrak{R}, \quad (91)$$

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & D_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

For a while we will consider these two cases together, assuming that

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c & d & e \\ 0 & 0 & z & x & 0 & f \\ 0 & 0 & 0 & z & 0 & g \\ 0 & 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad |z| = 1, \quad x \in \mathbb{C}.$$

Then the condition of the  $H$ -normality of  $N$  is equivalent to the system

$$a\bar{z} = z\bar{h} \tag{92}$$

$$a\bar{x} + b\bar{z} = x\bar{h} + z\bar{g} \tag{93}$$

$$2\Re\{a\bar{c}\} + |b|^2 + |d|^2 = 2\Re\{f\bar{h}\} + |g|^2 + |p|^2. \tag{94}$$

As is customary, we can assume that  $a = 1$ ,  $h = z^2$ . Let us use the ( $H$ -unitary) transformation

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2}|d|^2 & -d & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \bar{d} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It reduces  $N - \lambda I$  to the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & b' & c' & 0 & e' \\ 0 & 0 & z & x & 0 & f' \\ 0 & 0 & 0 & z & 0 & g' \\ 0 & 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 & p' \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Further, take the transformation

$$T = \begin{pmatrix} 1 & z\bar{g}' & \bar{z}c' - x\bar{g}' & 0 & 0 & -\frac{1}{2}|\bar{z}c' - x\bar{g}'|^2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -z\bar{c}' + \bar{x}g' \\ 0 & 0 & 0 & 1 & 0 & -\bar{z}g' \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and carry the matrix  $N - \lambda I$  into the form

$$N - \lambda I = \begin{pmatrix} 0 & 1 & b'' & 0 & 0 & e'' \\ 0 & 0 & z & x & 0 & f'' \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 & p'' \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now note that  $p'' \neq 0$  because otherwise  $v_5 \in S_0$ . Since the rotation of the vector  $v_5$  about any angle does not change the matrix  $H$ , we can assume that  $p'' = r_2 \in \Re > 0$  (we put  $\tilde{v}_5 = e^{i \arg p''} v_5$ ). The transformation

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & e''/r_2 & -\frac{1}{2}|e''/r_2|^2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\bar{e}''/r_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

reduces the matrix  $N - \lambda I$  to the final form:

$$N - \lambda I = \begin{pmatrix} 0 & 1 & b''' & 0 & 0 & 0 \\ 0 & 0 & z & x & 0 & f''' \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 & r_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now we will distinguish the cases (90) and (91).

(a)  $z = 1, x \in \mathfrak{R}$ . According to conditions (93) and (94) of the  $H$ -normality of  $N$ ,

$$N - \lambda I = \begin{pmatrix} 0 & 1 & 2ir_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & ir_1 & 0 & 2r_1^2 - r_2^2/2 + ir_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & r_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{array}{l} r_1, r_2, r_3 \in \mathfrak{R}, \\ r_2 > 0. \end{array}$$

Let us show that  $r_1, r_2, r_3$  are  $H$ -unitary invariants. Indeed, suppose some matrix  $T$  satisfies conditions (37)  $TT^{[*]} = I$  and (36)  $(N - \lambda I)T = T(\tilde{N} - \lambda I)$ , where

$$\tilde{N} - \lambda I = \begin{pmatrix} 0 & 1 & 2i\tilde{r}_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & i\tilde{r}_1 & 0 & 2\tilde{r}_1^2 - \tilde{r}_2^2/2 + i\tilde{r}_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \tilde{r}_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{array}{l} \tilde{r}_1, \tilde{r}_2, \tilde{r}_3 \in \mathfrak{R}, \\ \tilde{r}_2 > 0. \end{array}$$

From (36) it follows that

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \\ 0 & t_{11} & t_{23} & t_{24} & 0 & t_{26} \\ 0 & 0 & t_{11} & t_{34} & 0 & t_{36} \\ 0 & 0 & 0 & t_{11} & 0 & t_{46} \\ 0 & 0 & 0 & t_{54} & t_{55} & t_{56} \\ 0 & 0 & 0 & 0 & 0 & t_{11} \end{pmatrix}.$$

Using (87), we get:  $t_{54} = 0, |t_{11}| = 1$ . As above (see the argument before Lemma 5), we can assume that  $t_{11} = 1$ . Then  $t_{34} = -\overline{t_{23}}$  (condition (87)) and  $i(\tilde{r}_1 - r_1) = t_{34} - t_{23}$  (condition (82)), hence,  $\tilde{r}_1 = r_1$  and  $\mathcal{R}et_{23} = 0$ . Further, from (83) it follows that  $r_2 = r_2 t_{55}$ , from (87) that  $|t_{55}| = 1$ . As  $r_2, \tilde{r}_2 > 0, \tilde{r}_2 = r_2$  and  $t_{55} = 1$ . Thus,

$$T = \begin{pmatrix} 1 & it & t_{24} & 0 \\ 0 & 1 & it & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad t \in \mathfrak{R}, \quad 2\mathcal{R}et_{24} + t^2 = 0.$$

Substituting  $T_4$  in (80), we get  $t_{12} = it, t_{13} = t_{24} - r_1 t$ ; replacing  $T_5$  by  $-T_4 H_1 T_2^*$  in (83), we have  $i\tilde{r}_3 = ir_3 - 2\mathcal{R}et_{24} - t^2$ , hence  $\tilde{r}_3 = r_3$ . This completes the proof of the  $H$ -unitary invariance of  $r_1, r_2, r_3$ .

(b)  $0 < \arg z < \pi, x \in \mathfrak{R}$ . Applying the condition of the  $H$ -normality, we get

$$N - \lambda I = \begin{pmatrix} 0 & 1 & -2ir_1 \mathcal{I}m z & 0 & 0 & 0 \\ 0 & 0 & z & r_1 & 0 & (2r_1^2 \mathcal{I}m^2 z - r_2^2/2 + ir_3)z^2 \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 & r_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where  $r_1, r_2, r_3 \in \mathfrak{R}, r_2 > 0$ . That the numbers  $z, r_1, r_2, r_3$  are  $H$ -unitary invariants can be checked as in (a) above. That the forms obtained are not  $H$ -unitary similar can also be checked by the reader by using formulas (80) - (87).

Because of Proposition 2 the forms obtained are indecomposable so that we have proved the following lemma:

**Lemma 6** *If an indecomposable  $H$ -normal operator  $N$  ( $N : C^6 \rightarrow C^6$ ) has the only eigenvalue  $\lambda$ ,  $\dim S_0 = 1$ , the internal operator  $N_1$  is decomposable, then the pair  $\{N, H\}$  is unitarily similar to one and only one of canonical pairs  $\{(18), (20)\}, \{(19), (20)\}$ :*

$$N = \begin{pmatrix} \lambda & 1 & 2ir_1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & ir_1 & 0 & 2r_1^2 - r_2^2/2 + ir_3 \\ 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 1 \\ 0 & 0 & 0 & 0 & \lambda & r_2 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \quad r_1, r_2 \in \mathfrak{R}, r_2 > 0,$$

$$N = \begin{pmatrix} \lambda & 1 & -2ir_1 \mathcal{I}m z & 0 & 0 & 0 \\ 0 & \lambda & z & r_1 & 0 & (2r_1^2 \mathcal{I}m^2 z - r_2^2/2 + ir_3)z^2 \\ 0 & 0 & \lambda & z & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & z^2 \\ 0 & 0 & 0 & 0 & \lambda & r_2 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$|z| = 1, 0 < \arg z < \pi, r_1, r_2, r_3 \in \mathfrak{R}, r_2 > 0,$$

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & D_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where  $z, r_1, r_2, r_3$  are  $H$ -unitary invariants.

#### 5.2.4 $n = 7$

We will show that this alternative is impossible. Indeed, if  $\dim V_2 = 4, \dim V_3 = 1$ , then, in accordance with Theorem 1 of [1],

$$N - \lambda I = \begin{pmatrix} 0 & a & b & c & d & e & f \\ 0 & 0 & \cos \alpha & \sin \alpha & 0 & 0 & g \\ 0 & 0 & 0 & 0 & 1 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad 0 < \alpha \leq \pi/2,$$

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the conditions of the  $H$ -normality of  $N$  are as follows:

$$\begin{aligned} a &= \bar{q} \cos \alpha \\ 0 &= \bar{q} \sin \alpha \\ b \cos \alpha + c \sin \alpha &= \bar{h} \\ 2\mathcal{R}e\{a\bar{d}\} + |b|^2 + |c|^2 + |e|^2 &= 2\mathcal{R}e\{g\bar{q}\} + |h|^2 + |p|^2 + |r|^2. \end{aligned}$$

Since  $\sin\alpha \neq 0$ ,  $q = 0$ , hence  $a = 0$ . Thus,  $(N - \lambda I)v_2 = (N^{[*]} - \bar{\lambda}I)v_2 = 0$  which contradicts the condition  $S_0 \cap S = \{0\}$ .

Thus, we have classified all indecomposable operators with one-dimensional subspace  $S_0$ . Now let us consider the case when  $\dim S_0 = 2$ .

### 5.3 $\dim S_0 = 2$

Let  $S_0$  be 2-dimensional. Since the operator  $H_1 = H|_{S_0}$  has only positive eigenvalues, one can assume that  $H_1 = I$ .  $N_1$  is a usual normal operator having the only eigenvalue  $\lambda$ , hence,  $N_1 = \lambda I$ . As a result, we have

$$N = \begin{pmatrix} \lambda I & N_1 & N_2 \\ 0 & \lambda I & N_3 \\ 0 & 0 & \lambda I \end{pmatrix}, \quad (95)$$

$$H = \begin{pmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{pmatrix}. \quad (96)$$

Below we will not stipulate that the pair  $\{N, H\}$  has form  $\{(95), (96)\}$ .

For  $N$  to be  $H$ -normal it is necessary and sufficient to have

$$N_1 N_1^* = N_3^* N_3. \quad (97)$$

According to Theorem 1, for indecomposable operators  $n \leq 8$ . Let us consider the cases  $n = 4, 5, 6, 7, 8$  one after another.

#### 5.3.1 $n = 4$

In this case  $C^4 = S_0 \dot{+} S_1$ ,

$$N - \lambda I = \begin{pmatrix} 0 & N_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Condition (97) of the  $H$ -normality of  $N$  does not restrict the submatrix  $N_2$  (its terms  $a, b, c, d$ ). If  $N_2 = 0$ , the operator  $N$  is decomposable because the nondegenerate subspace  $V = \text{span}\{v_1, v_3\}$  is invariant for  $N$  and  $N^{[*]}$ . Thus,  $N_2$  can be either of rank 1 or of rank 2 ( $\text{rg } N_2 = 1$  or  $2$ ).

(a)  $\text{rg } N_2 = 2$ . Suppose an  $H$ -unitary transformation  $T$

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

reduces  $N - \lambda I$  to the form  $\tilde{N} - \lambda I$ :

$$N - \lambda I = \begin{pmatrix} 0 & N_2 \\ 0 & 0 \end{pmatrix}, \quad \tilde{N} - \lambda I = \begin{pmatrix} 0 & \widetilde{N}_2 \\ 0 & 0 \end{pmatrix}.$$

Then conditions (98) - (100) must be satisfied:

$$N_2 T_3 = 0 \quad (98)$$

$$N_2 T_4 = T_1 \widetilde{N}_2 \quad (99)$$

$$0 = T_3 \widetilde{N}_2. \quad (100)$$

Since  $N_2$  is invertible, (98) holds only if  $T_3 = 0$ . Hence,  $T$  is  $H$ -unitary iff

$$T_1 T_4^* = I \quad (101)$$

$$T_1 T_2^* + T_2 T_1^* = 0. \quad (102)$$

From system (101) - (102) it follows that without loss of generality we can consider only block diagonal transformations of the form  $T = T_1 \oplus T_1^{*-1}$  because  $T_2$  does not figure in equations (98) - (100).

Thus, the only condition (99)  $N_2 = T_1 \widetilde{N}_2 T_1^*$  must be satisfied. Applying Proposition 3 from Appendix, we obtain that the submatrix  $N_2$  can be reduced to one of the canonical forms

$$N_2 = \begin{pmatrix} z & \varrho e^{-i\pi/3} z \\ 0 & e^{i\pi/3} z \end{pmatrix}, \quad N_2 = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix},$$

where  $z, z_1, z_2, \varrho$  ( $|z| = 1, \varrho \in \Re \geq \sqrt{3}, 0 \leq \arg z < \pi$  if  $\varrho > \sqrt{3}, |z_1| = |z_2| = 1, \arg z_1 \leq \arg z_2$ ) are invariants. For the latter form the operator  $N$  is decomposable because the nondegenerate subspace  $V = \text{span}\{v_1, v_3\}$  is invariant both for  $N$  and  $N^{[*]}$ . For the former we obtain the following canonical form:

$$N - \lambda I = \begin{pmatrix} 0 & 0 & z & r e^{-i\pi/3} z \\ 0 & 0 & 0 & e^{i\pi/3} z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{array}{l} |z| = 1, r \in \Re \geq \sqrt{3}, \\ 0 \leq \arg z < \pi \text{ if } r > \sqrt{3}. \end{array}$$

(b)  $rg N_2 = 1$ . Then

$$N_2 = \begin{pmatrix} ka & kb \\ la & lb \end{pmatrix}, \quad |a| + |b| \neq 0, |k| + |l| \neq 0.$$

If  $\bar{a} = k\bar{b}$ , then  $v = bv_3 - av_4 \neq 0$  belongs both to  $S_0$  and  $S_1$ , which is impossible ( $S_0 \cap S_1 = \{0\}$ ). Thus, we can assume that  $\bar{a} \neq k\bar{b}$ . Taking the transformation  $T = T_1 \oplus T_1^{*-1}$ , where

$$T_1 = \begin{pmatrix} \bar{a} & k \\ \bar{b} & l \end{pmatrix},$$

we obtain one more canonical form:

$$N - \lambda I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

**Lemma 7** *If an indecomposable  $H$ -normal operator  $N$  ( $N : C^4 \rightarrow C^4$ ) has the only eigenvalue  $\lambda$ ,  $\dim S_0 = 2$ , then the pair  $\{N, H\}$  is unitarily similar to one and only one of canonical pairs  $\{(21), (23)\}, \{(22), (23)\}$ :*

$$N = \begin{pmatrix} \lambda & 0 & z & r e^{-i\pi/3} z \\ 0 & \lambda & 0 & e^{i\pi/3} z \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \begin{array}{l} |z| = 1, r \in \Re \geq \sqrt{3}, \\ 0 \leq \arg z < \pi \text{ if } r > \sqrt{3}, \end{array}$$

$$N = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},$$

where  $r, z$  are  $H$ -unitary invariants.

**Proof:** The possibility to reduce  $N$  to one of forms (21), (22) is proved before the lemma. The argument in (a) above shows that these forms are not similar, hence, they are not  $H$ -unitarily similar. Thus, we must only prove the indecomposability of  $N$ .

Show that the first canonical form is indecomposable. Assume the converse. Let some nondegenerate subspace  $V$  be invariant for  $N$  and  $N^{[*]}$ . Then there exists a nonzero vector  $w_1 \in V : w_1 \in S_0$ . Therefore,  $\exists w_2 = av_3 + bv_4 + v \in V$  ( $v \in S_0, |a| + |b| \neq 0$ ).

$$\begin{aligned}(N - \lambda I)w_2 &= azv_1 + b(re^{-i\pi/3}zv_1 + e^{i\pi/3}zv_2), \\ (N^{[*]} - \bar{\lambda}I)w_2 &= a(\bar{z}v_1 + re^{i\pi/3}\bar{z}v_2) + be^{-i\pi/3}\bar{z}v_2.\end{aligned}$$

Since  $\min\{\dim V, \dim V^{[\perp]}\} \leq 2$ , it can be assumed that  $\dim V \leq 2$ . As the vectors  $w_1$  and  $w_2$  are linearly independent, we get  $\dim V = 2$ . Therefore, the vectors  $(N - \lambda I)w_2$  and  $(N^{[*]} - \bar{\lambda}I)w_2$  must be linearly dependent, i.e., the following condition must be satisfied:

$$(a + bre^{-i\pi/3})(are^{i\pi/3} + be^{-i\pi/3}) = abe^{i\pi/3}. \quad (103)$$

Since (103) breaks if either  $a$  or  $b$  is equal to zero, we can rewrite (103) as follows:

$$\left(\frac{a}{b}\right)^2 re^{i\pi/3} + \left(\frac{a}{b}\right)(e^{-i\pi/3} - e^{i\pi/3} + r^2) + re^{-2i\pi/3} = 0. \quad (104)$$

Discriminant of (104) is equal to  $r^4 - 2r^2 - 3$ . Since  $r^2 \geq 3$ , it is nonnegative. Therefore,

$$\frac{a}{b} = \frac{i\sqrt{3} - r^2 \pm \sqrt{r^4 - 2r^2 - 3}}{r(1 + i\sqrt{3})}.$$

Consequently,  $|\frac{a}{b}|^2 = \frac{1}{2}(r^2 - 1 \mp \sqrt{r^4 - 2r^2 - 3})$ , therefore,  $[w_2, (N - \lambda I)w_2] = z|b|^2(|\frac{a}{b}|^2 + \frac{a}{b}re^{i\pi/3} + e^{-i\pi/3}) = 0$ . Thus, the subspace  $V$  is degenerate, i.e., the operator  $N$  is indecomposable.

For the second matrix  $N$  we see that the vectors  $(N - \lambda I)w_2$  and  $(N^{[*]} - \bar{\lambda}I)w_2$  ( $w_2 = av_3 + bv_4 + v, v \in S_0$ ) can be linearly dependent only if  $a = b = 0$ . Therefore,  $N$  is also indecomposable. This concludes the proof of the lemma.

### 5.3.2 $n = 5$

The matrix  $N - \lambda I$  has the form

$$N - \lambda I = \begin{pmatrix} 0 & N_1 & N_2 \\ 0 & 0 & N_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a & c & d \\ 0 & 0 & b & e & f \\ 0 & 0 & 0 & g & h \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so that condition (97) of the  $H$ -normality of  $N$  amounts to the system

$$\begin{aligned}|a| &= |g| \\ a\bar{b} &= \bar{g}h \\ |b| &= |h|.\end{aligned}$$

The latter means that  $g = \bar{a}z, h = \bar{b}z$  ( $|z| = 1$ ). Note that  $a$  and  $b$  are not equal to zero simultaneously because otherwise  $v_3 \in S_0$ , which is impossible.

Take the transformation  $T = T_1 \oplus I \oplus T_1^{*-1}$ , where

$$T_1 = \begin{pmatrix} a & t_{12} \\ b & t_{22} \end{pmatrix}, \quad at_{22} \neq bt_{12},$$

and reduce  $N - \lambda I$  to the form

$$N - \lambda I = \begin{pmatrix} 0 & 0 & 1 & c' & d' \\ 0 & 0 & 0 & e' & f' \\ 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad |z| = 1.$$

Now we fix the form of the submatrices  $N_1$  and  $N_3$  so that the following transformations will change only the submatrix  $N_2$ . At first, apply the transformation

$$T = \begin{pmatrix} I & T_2 & -\frac{1}{2}T_2T_2^* \\ 0 & I & -T_2^* \\ 0 & 0 & I \end{pmatrix}, \quad (105)$$

where  $T_2^* = (0 \ d')$ , and reduce  $N_2$  to the form

$$N_2 = \begin{pmatrix} c'' & 0 \\ e'' & f'' \end{pmatrix}.$$

Now let us consider two cases:  $f'' = 0$  and  $f'' \neq 0$ .

(a)  $f'' = 0$ . Then  $e'' \neq 0$  because otherwise  $v_5 \in S_0$ . Subjecting  $N - \lambda I$  to the transformation  $T = T_1 \oplus I \oplus T_1^{*-1}$ , where

$$T_1 = \begin{pmatrix} 1 & c'' \\ 0 & e'' \end{pmatrix},$$

we get

$$N_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(b)  $f'' \neq 0$ . Then one can assume that  $|f''| = 1$  (to this end it is sufficient to put  $\tilde{v}_2 = \sqrt{|f''|}v_2$ ,  $\tilde{v}_5 = v_5/\sqrt{|f''|}$ ). Thus,  $f'' = z_1$ ,  $|z_1| = 1$ .

If  $z_1^2 \neq z$ , then  $N$  is decomposable. Indeed, applying

$$T = \begin{pmatrix} T_1 & -T_1T_5^* & -\frac{1}{2}T_1T_5^*T_5 \\ 0 & I & T_5 \\ 0 & 0 & T_1^{*-1} \end{pmatrix}, \quad (106)$$

where

$$T_1 = \begin{pmatrix} 1 & z_1\bar{e}''/(1-\bar{z}z_1^2) \\ 0 & 1 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 0 & z_1^2\bar{e}''/(1-\bar{z}z_1^2) \end{pmatrix},$$

we reduce  $N_2$  to the diagonal form  $N_2 = c''' \oplus z_1$ . Now the nondegenerate subspace  $V = \text{span}\{v_2, v_5\}$  is invariant for  $N$  and  $N^{[*]}$ , hence,  $N$  is decomposable.

Let  $z_1^2 = z$ . Note that if  $e'' = 0$ , then  $N$  is decomposable ( $V = \text{span}\{v_2, v_5\}$  is nondegenerate,  $NV \subseteq V$ ,  $N^{[*]}V \subseteq V$ ). Thus,  $e'' \neq 0$ . Taking transformation (106) with

$$T_1 = \begin{pmatrix} 1 & iz_1c_2''/|e''| \\ 0 & e^{i \arg e''} \end{pmatrix}, \quad T_5 = \begin{pmatrix} -z_1(c_1'' + c_2''^2/|e''|^2)/2 & iz_1^2c_2''/|e''| \end{pmatrix},$$

where  $c_1'' = \text{Re}\{c''\bar{z}_1\}$ ,  $c_2'' = \text{Im}\{c''\bar{z}_1\}$ , we reduce  $N_2$  to the final form

$$N_2 = \begin{pmatrix} 0 & 0 \\ r & z_1 \end{pmatrix}, \quad r = |e''| > 0.$$

**Lemma 8** *If an indecomposable  $H$ -normal operator  $N$  ( $N : C^5 \rightarrow C^5$ ) has the only eigenvalue  $\lambda$ ,  $\dim S_0 = 2$ , then the pair  $\{N, H\}$  is unitarily similar to one and only one of canonical pairs  $\{(24), (26)\}$ ,  $\{(25), (26)\}$ :*

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 \\ 0 & 0 & \lambda & z & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \quad |z| = 1,$$

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & r & z \\ 0 & 0 & \lambda & z^2 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \quad |z| = 1, \quad r \in \Re > 0,$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_1 & 0 \\ I_2 & 0 & 0 \end{pmatrix},$$

where  $z, r$  are  $H$ -unitary invariants.

**Proof:** The possibility to reduce  $N$  to one of forms (24), (25) is proved before the lemma. Hence, it is necessary to show that these forms are indecomposable, are not  $H$ -unitarily similar to each other and their terms  $z, r$  are  $H$ -unitary invariants. These statements may be proved as follows.

For the block triangular matrix

$$T = \begin{pmatrix} T_1 & T_2 & T_3 \\ 0 & T_4 & T_5 \\ 0 & 0 & T_6 \end{pmatrix} \quad (107)$$

to satisfy condition (36)  $NT = T\tilde{N}$ , where

$$N - \lambda I = \begin{pmatrix} 0 & N_1 & N_2 \\ 0 & 0 & N_3 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{N} - \lambda I = \begin{pmatrix} 0 & \tilde{N}_1 & \tilde{N}_2 \\ 0 & 0 & \tilde{N}_3 \\ 0 & 0 & 0 \end{pmatrix},$$

it is necessary and sufficient to have

$$N_1 T_4 = T_1 \tilde{N}_1 \quad (108)$$

$$N_1 T_5 + N_2 T_6 = T_1 \tilde{N}_2 + T_2 \tilde{N}_3 \quad (109)$$

$$N_3 T_6 = T_4 \tilde{N}_3. \quad (110)$$

If  $H$  has form (96), then for (107) to be  $H$ -unitary it is necessary and sufficient to have

$$T_1 T_6^* = I \quad (111)$$

$$T_1 T_5^* + T_2 T_4^* = 0 \quad (112)$$

$$T_1 T_3^* + T_2 T_2^* + T_3 T_1^* = 0 \quad (113)$$

$$T_4 T_4^* = I. \quad (114)$$

If an  $H$ -unitary transformation  $T$  reduces matrix (25) (the second) to form (24) (the first), then from Corollary of Proposition 1 it follows that  $T$  has block form (107) and, according to (36),

$$T_1 = \begin{pmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{pmatrix}. \quad (115)$$

Apply condition (109), replacing  $T_6$  by  $T_1^{*-1}$  (111),  $T_2$  by  $-T_1 T_5^* T_4$  (112). Then we get:  $z/\overline{t_{22}} = 0$ . This contradiction proves that the canonical forms are not  $H$ -unitarily similar.

If

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r & z \\ 0 & 0 & 0 & z^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix} T = T \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \tilde{r} & \tilde{z} \\ 0 & 0 & 0 & \tilde{z}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$|z| = |\tilde{z}| = 1$ ,  $r, \tilde{r} \in \mathfrak{R} > 0$ , then  $T$  has form (107), the submatrix  $T_1$  having form (115) and  $t_{11} = t_{33}$ . Since  $|t_{33}| = 1$  (condition (114)), we can assume that  $t_{11} = t_{33} = 1$ . Replace  $T_6$  by  $T_1^{*-1}$  and apply (110); we have  $\tilde{z}^2 = z^2$ . Now substitute  $T_1^{*-1}$  for  $T_6$  and  $-T_1 T_5^*$  for  $T_2$  in (109). We obtain

$$t_{35} = \tilde{z}t_{12} \quad (116)$$

$$r - z\overline{t_{12}/t_{22}} = \tilde{r}t_{22} - z^2 t_{22}\overline{t_{35}} \quad (117)$$

$$z/\overline{t_{22}} = \tilde{z}t_{22}. \quad (118)$$

From (118) it follows that  $|t_{22}| = 1$ ,  $\tilde{z} = z$ . Hence,  $1/\overline{t_{22}} = t_{22}$ ,  $t_{35} = zt_{12}$  and  $r = \tilde{r}t_{22}$ . Therefore,  $r = \tilde{r}|t_{22}|$ , i.e.,  $\tilde{r} = r$ . Thus, the numbers  $z, r$  are  $H$ -unitary invariants of canonical form (25). That  $z$  is an  $H$ -unitary invariant of (24) can be checked in the similar way.

There remains to prove that matrices (24) and (25) are indecomposable. The proof is by reductio ad absurdum. Suppose some nondegenerate subspace  $V$  is invariant for  $N$  and  $N^{[*]}$  ( $N$  has form (24)). As  $\min\{\dim V, \dim V^{\perp}\} \leq 2$ , we can assume that  $\dim V \leq 2$ . Since there exists a vector  $w_1 \neq 0 \in S_0$  :  $w_1 \in V$ , there exists also a vector  $w_2 = av_3 + bv_4 + cv_5 + v \in V$  ( $v \in S_0$ ,  $|b| + |c| \neq 0$ ). As the vectors  $(N - \lambda I)w_2 = av_1 + b(v_2 + zv_3)$  and  $(N^{[*]} - \bar{\lambda}I)w_2 = a\bar{z}v_1 + bv_3 + cv_1$  must be linearly dependent, we obtain  $b = 0$ . But in this case the subspace  $V$  will be degenerate because  $[(N - \lambda I)w_2, w_2] = 0$ . This contradiction proves the indecomposability of (24). Now let us check the indecomposability of (25). Suppose a nondegenerate subspace  $V$  is invariant both for  $N$  and  $N^{[*]}$ . Then, as before,  $\exists w_1 \neq 0 \in S_0$  :  $w_1 \in V$  and  $\exists w_2 = av_3 + bv_4 + cv_5 + v \in V$  ( $v \in S_0$ ,  $|b| + |c| \neq 0$ ). Therefore, the vectors  $(N - \lambda I)w_2 - z^2(N^{[*]} - \bar{\lambda}I)w_2 = brv_2 - crz^2v_1$  and  $(N - \lambda I)w_2 = av_1 + brv_2 + bz^2v_3 + czv_2$  have to be linearly dependent. Hence,  $b = 0 \Rightarrow c = 0$ . The contradiction obtained proves that (25) is also indecomposable. The proof of the lemma is completed.

### 5.3.3 $n = 6$

The matrix  $N - \lambda I$  has the form

$$N - \lambda I = \begin{pmatrix} 0 & N_1 & N_2 \\ 0 & 0 & N_3 \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } N_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The submatrix  $N_1$  is not equal to zero because then condition (97) of the  $H$ -normality of  $N$  implies  $N_3 = 0$  so that  $v_3, v_4 \in S_0$ , which is impossible. Thus, we must consider two alternatives:  $rg N_1 = 2$  and  $rg N_1 = 1$ .

(a)  $rg N_1 = 2$ . At first apply the transformation  $T = N_1 \oplus I \oplus N_1^{*-1}$ ; it takes  $N_1$  to  $I$ . Since  $N_1$  has become equal to  $I$ ,  $N_3$ , according to (97), has become unitary. Recall that any unitary matrix is unitarily similar to some diagonal one with nonzero terms of modulus 1; moreover, this representation is unique to within order of diagonal terms. Thus,  $\exists U (UU^* = I) : \widetilde{N}_3 = U^* N_3 U$ , where

$$\widetilde{N}_3 = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}, |z_1| = |z_2| = 1, \arg z_1 \leq \arg z_2. \quad (119)$$

If we subject  $N - \lambda I$  to the transformation  $T = U \oplus U \oplus U$ , then  $N_3$  maps to (119) and  $N_1 = I$  does not change.

Note that if  $z_1 \neq z_2$ ,  $N$  is decomposable. To check this it is sufficient to reduce

$$N_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \quad (120)$$

to the diagonal form by means of transformation (105) with the submatrix

$$T_2 = \begin{pmatrix} 0 & (\bar{g} - \bar{z}_1 f)/(1 - \bar{z}_1 z_2) \\ (\bar{f} - \bar{z}_2 g)/(1 - z_1 \bar{z}_2) & 0 \end{pmatrix}$$

(this transformation does not change  $N_1$  and  $N_3$ ). Now the nondegenerate subspace  $V = \text{span}\{v_1, v_3, v_5\}$  is invariant for  $N$  and  $N^{[*]}$ , hence,  $N$  is decomposable.

Thus, for  $N$  to be indecomposable  $N_3$  must be equal to  $zI$ . Show that in case when  $z = -1$   $N$  is also decomposable. Indeed, apply the transformation

$$T = \begin{pmatrix} U & -\frac{1}{2}N_2U & -\frac{1}{8}N_2N_2^*U \\ 0 & U & \frac{1}{2}N_2^*U \\ 0 & 0 & U \end{pmatrix},$$

where  $U$  is a unitary matrix reducing  $N_2 + N_2^*$  to the diagonal form ( $U$  is known to exist). Then  $N_2$  becomes diagonal; we already know that in this case  $N$  is decomposable.

Thus,  $N = zI$ ,  $z \neq -1$ . Now we will apply only transformations preserving the submatrices  $N_1$  and  $N_3$ . First let us take (105) with

$$T_2 = \begin{pmatrix} 0 & 0 \\ \frac{0}{f} & 0 \end{pmatrix}$$

and carry submatrix (120) to the form

$$N_2 = \begin{pmatrix} e' & 0 \\ g' & h' \end{pmatrix}.$$

Further, apply transformation (105) with

$$T_2 = \begin{pmatrix} t_{13} & 0 \\ 0 & t_{24} \end{pmatrix},$$

where  $\mathcal{R}e\{\overline{t_{13}} + zt_{13}\} = \mathcal{R}e e'$ ,  $\mathcal{R}e\{\overline{t_{24}} + zt_{24}\} = \mathcal{R}e h'$  (since  $z \neq -1$ , these equations are solvable for any  $e'$  and  $h'$ ). After this transformation

$$N_2 = \begin{pmatrix} ir_1 & 0 \\ g' & ir_2 \end{pmatrix}.$$

One can assume that  $g' = r_3 \in \mathfrak{R} \geq 0$ . To this end it is sufficient to put  $\tilde{v}_2 = e^{i \arg g'} v_2$ ,  $\tilde{v}_4 = e^{i \arg g'} v_4$ ,  $\tilde{v}_6 = e^{i \arg g'} v_6$ . Now apply the transformation

$$T = \begin{pmatrix} T_1 & T_1T_2 & -\frac{1}{2}T_1T_2T_2^* \\ 0 & T_1 & -T_1T_2^* \\ 0 & 0 & T_1 \end{pmatrix}, \text{ where}$$

$$T_1 = 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ -(z+1)/|z+1| & (z+1)/|z+1| \end{pmatrix},$$

$$T_2 = \frac{1}{2} \begin{pmatrix} -r_3/|z+1| & 0 \\ (ir_2 - ir_1) - r_3(z+1)/|z+1| & r_3/|z+1| \end{pmatrix}.$$

We get:

$$N_2 = \begin{pmatrix} ir_1' & 0 \\ g'' & ir_1' \end{pmatrix}, \quad r_1' = \frac{1}{2}(r_1 + r_2).$$

As above, we can assume that  $g'' \in \mathfrak{R} \geq 0$ . For  $N$  to be indecomposable  $g''$  must be nonzero so that  $g'' > 0$ . This is the final form of the matrix  $N - \lambda I$ :

$$N - \lambda I = \begin{pmatrix} 0 & 0 & 1 & 0 & ir_1 & 0 \\ 0 & 0 & 0 & 1 & r_2 & ir_1 \\ 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{matrix} |z| = 1, z \neq -1, \\ r_1, r_2 \in \mathfrak{R}, r_2 > 0. \end{matrix} \quad (121)$$

Let us show that  $z, r_1, r_2$  are  $H$ -unitary invariants. To this end suppose that an  $H$ -unitary matrix  $T$  reduces (121) to the form

$$\tilde{N} - \lambda I = \begin{pmatrix} 0 & I & \tilde{N}_2 \\ 0 & 0 & \tilde{z}I \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{N}_2 = \begin{pmatrix} i\tilde{r}_1 & 0 \\ \tilde{r}_2 & i\tilde{r}_1 \end{pmatrix}, \quad \begin{matrix} |\tilde{z}| = 1, \tilde{z} \neq -1, \\ \tilde{r}_1, \tilde{r}_2 \in \mathfrak{R}, \tilde{r}_2 > 0. \end{matrix}$$

By Corollary of Proposition 1,  $T$  must have block triangular form (107), therefore, systems (108) - (110) and (111) - (114) must hold. From (108), (114), and (111) it follows that  $T_1 = T_4 = T_6 = T_6^{*-1}$ . Now from (110) it follows that  $\tilde{z} = z$ . Combining (112) and (109), we get  $N_2 = T_1 \tilde{N}_2 T_1^* + z T_2 T_1^* + T_1 T_2^*$ . If we denote

$$T_2' = T_2 T_1^* = \begin{pmatrix} t'_{11} & t'_{12} \\ t'_{21} & t'_{22} \end{pmatrix}$$

and write out the general form for  $2 \times 2$  unitary matrix

$$T_1 = \begin{pmatrix} \varrho s_1 & \sqrt{1 - \varrho^2} s_2 \\ \sqrt{1 - \varrho^2} s_3 & -\varrho \overline{s_1} s_2 s_3 \end{pmatrix}, \quad \varrho \in [0, 1], \quad |s_1| = |s_2| = |s_3| = 1, \quad (122)$$

then we obtain

$$\begin{aligned} i r_1 &= i \tilde{r}_1 + \varrho \sqrt{1 - \varrho^2} \overline{s_1} s_2 \tilde{r}_2 + z t'_{11} + \overline{t'_{11}} \\ i r_1 &= i \tilde{r}_1 - \varrho \sqrt{1 - \varrho^2} \overline{s_1} s_2 \tilde{r}_2 + z t'_{22} + \overline{t'_{22}}. \end{aligned}$$

Summing these equalities, we get

$$2i r_1 = 2i \tilde{r}_1 + z t'_{11} + \overline{t'_{11}} + z t'_{22} + \overline{t'_{22}}.$$

It is easy to check that if  $\mathcal{R}e\{zt + \bar{t}\} = 0$  ( $z \neq -1$ ), then  $\mathcal{I}m\{zt + \bar{t}\} = 0$ . In our case  $t'_{11} + t'_{22}$  plays the role of  $t$ , therefore, we have  $z t'_{11} + \overline{t'_{11}} + z t'_{22} + \overline{t'_{22}} = 0$ . Hence  $\tilde{r}_1 = r_1$ . Let us check that from the obtained equality  $\tilde{r}_1 = r_1$  it follows that  $\tilde{r}_2 = r_2$ . Indeed,  $z N_2^* - N_2 = T_1 (z \tilde{N}_2^* - \tilde{N}_2) T_1^*$ .

$$z N_2^* - N_2 = \begin{pmatrix} -i r_1 (z + 1) & z r_2 \\ -r_2 & -i r_1 (z + 1) \end{pmatrix};$$

the determinant of  $z N_2^* - N_2$ , which does not change the similarity, is equal to  $-r_1^2 (z + 1)^2 + z r_2^2$ , hence  $r_2^2 = \tilde{r}_2^2$ . Since sign of  $r_2$  coincides with that of  $\tilde{r}_2$ ,  $\tilde{r}_2 = r_2$ . The proof of the  $H$ -unitary invariance of the numbers  $r_1, r_2$  is completed.

(b)  $rg N_1 = 1$ . Let us show that in this case  $N$  is decomposable. In fact,

$$N_1 = \begin{pmatrix} ka & kb \\ la & lb \end{pmatrix}, \quad |a| + |b| \neq 0, \quad |k| + |l| \neq 0.$$

Taking  $T = T_1 \oplus I \oplus T_1^{*-1}$ , where

$$T_1 = \begin{pmatrix} t_{11} & k \\ t_{21} & l \end{pmatrix}, \quad l t_{11} \neq k t_{21},$$

we reduce  $N_1$  to the form

$$N_1 = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}.$$

Without loss of generality one can assume that  $a \neq 0$  and, therefore, that  $a = 1$  (this may be achieved by putting  $\tilde{v}_2 = av_2, \tilde{v}_6 = v_6/\bar{a}$ ). If  $b \neq 0$ , apply the transformation  $T_1 \oplus T_4 \oplus T_1^{*-1}$ , where

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sqrt{|b|^2 + 1} \end{pmatrix}, \quad T_4 = \begin{pmatrix} 1/\sqrt{|b|^2 + 1} & |b|/\sqrt{|b|^2 + 1} \\ \bar{b}/\sqrt{|b|^2 + 1} & -e^{-i \arg b}/\sqrt{|b|^2 + 1} \end{pmatrix},$$

to the matrix  $N - \lambda I$  (we mean that  $a = 1$ ). Then we obtain

$$N_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

According to (97),

$$N_3 = \begin{pmatrix} 0 & z_1 \cos \alpha \\ 0 & z_2 \sin \alpha \end{pmatrix}, \quad |z_1| = |z_2| = 1, \quad 0 \leq \alpha \leq \pi/2.$$

Since  $v_4 \bar{\in} S_0$ ,  $\sin \alpha \neq 0$ . Therefore, we can apply the transformation  $T$  of form (105), where

$$T_2 = \begin{pmatrix} \bar{g} & (f - z_1 \bar{g} \cos \alpha)/(z_2 \sin \alpha) \\ 0 & 0 \end{pmatrix}$$

( $N_2$  has form (120)). Under the action of  $T$  the submatrices  $N_1$  and  $N_3$  do not change but the submatrix  $N_2$  becomes diagonal. Now the nondegenerate subspace  $V = \text{span}\{v_1, v_5\}$  is invariant for  $N$  and  $N^{[*]}$ , hence,  $N$  is decomposable.

**Lemma 9** *If an indecomposable  $H$ -normal operator  $N$  ( $N : C^6 \rightarrow C^6$ ) has the only eigenvalue  $\lambda$ ,  $\dim S_0 = 2$ , then the pair  $\{N, H\}$  is unitarily similar to canonical pair  $\{(27), (28)\}$ :*

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & ir_1 & 0 \\ 0 & \lambda & 0 & 1 & r_2 & ir_1 \\ 0 & 0 & \lambda & 0 & z & 0 \\ 0 & 0 & 0 & \lambda & 0 & z \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \begin{array}{l} |z| = 1, z \neq -1, \\ r_1, r_2 \in \mathfrak{R}, r_2 > 0, \end{array}$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_2 & 0 \\ I_2 & 0 & 0 \end{pmatrix},$$

where  $z, r_1, r_2$  are  $H$ -unitary invariants.

**Proof:** It is necessary to prove only the indecomposability of the canonical form because the rest was proved before the lemma. Suppose that a nondegenerate subspace  $V$  satisfies the conditions  $NV \subseteq V$ ,  $N^{[*]}V \subseteq V$ . As above, we can assume that  $\dim V \leq 3$  (see the proofs of the previous lemmas). Since  $\exists w_1 \neq 0 \in S_0 : w_1 \in V$ ,  $\exists w_2 = av_5 + bv_6 + v \in V$  ( $v \in (S_0 + S)$ ,  $|a| + |b| \neq 0$ ). The vectors  $(N - \lambda I)(N^{[*]} - \bar{\lambda} I)w_2 = av_1 + bv_2$  and  $(N - \lambda I - z(N^{[*]} - \bar{\lambda} I))w_2 = air_1(1 + z)v_1 - br_2zv_1 + bir_1(1 + z)v_2 + ar_2v_2$  must be linearly dependent because otherwise  $S_0 \subset V$  and  $\dim V \geq 4$ . Therefore,  $-b^2r_2z = a^2r_2$ . Since  $z \neq -1$ ,  $a = b = 0$ . This contradiction proves that  $N$  is indecomposable. The proof of the lemma is completed.

#### 5.3.4 $n = 7$

The matrix  $N - \lambda I$  has the form

$$N - \lambda I = \begin{pmatrix} 0 & N_1 & N_2 \\ 0 & 0 & N_3 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{where } N_1 = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}.$$

As in case when  $n = 6$ , one can check that  $N_1 \neq 0$ , therefore, we must consider the cases  $rg N_1 = 1$  and  $rg N_1 = 2$ . Show that the former alternative is also impossible. Indeed, if  $rg N_1 = 1$ , then

$$N_1 = \begin{pmatrix} ka & kb & kc \\ la & lb & lc \end{pmatrix}, \quad |a| + |b| + |c| \neq 0, \quad |k| + |l| \neq 0.$$

Applying the transformation  $T = T_1 \oplus I \oplus T_1^{*-1}$ , where

$$T_1 = \begin{pmatrix} t_{11} & k \\ t_{21} & l \end{pmatrix}, \quad lt_{11} \neq kt_{21},$$

we reduce  $N_1$  to the form

$$N_1 = \begin{pmatrix} 0 & 0 & 0 \\ a & b & c \end{pmatrix}.$$

Then from condition (97) of the  $H$ -normality of  $N$  it follows that

$$N_3 = \begin{pmatrix} 0 & s \\ 0 & u \\ 0 & w \end{pmatrix}.$$

Since there exists a nontrivial solution  $\{\alpha_i\}_1^3$  of the system

$$\begin{aligned} a\alpha_1 + b\alpha_2 + c\alpha_3 &= 0 \\ \bar{s}\alpha_1 + \bar{u}\alpha_2 + \bar{w}\alpha_3 &= 0, \end{aligned}$$

the nonzero vector  $v = \alpha_1v_3 + \alpha_2v_4 + \alpha_3v_5$  belongs to  $S_0$ , which contradicts the condition  $S_0 \cap S = \{0\}$ .

Thus,  $rg N_1 = 2$ . Then without loss of generality it can be assumed that

$$\det \begin{pmatrix} a & b \\ d & e \end{pmatrix} \neq 0.$$

Take the block diagonal transformation  $T_1 \oplus I \oplus T_1^{*-1}$ , where

$$T_1 = \begin{pmatrix} a & b \\ d & e \end{pmatrix}.$$

It reduces  $N_1$  to the form

$$N_1 = \begin{pmatrix} 1 & 0 & c' \\ 0 & 1 & f' \end{pmatrix}.$$

Further, apply the transformation  $T_1 \oplus T_2 \oplus T_1^{*-1}$ , where

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1+|f'|^2} \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{1+|f'|^2} & -f'/\sqrt{1+|f'|^2} \\ 0 & \bar{f}'/\sqrt{1+|f'|^2} & 1/\sqrt{1+|f'|^2} \end{pmatrix}.$$

Then we get

$$N_1 = \begin{pmatrix} 1 & b'' & c'' \\ 0 & 1 & 0 \end{pmatrix}.$$

Now take  $T = T_1 \oplus T_2 \oplus T_1^{*-1}$ , where

$$T_1 = \begin{pmatrix} \sqrt{1+|c''|^2} & b'' \\ 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1/\sqrt{1+|c''|^2} & 0 & -c''/\sqrt{1+|c''|^2} \\ 0 & 1 & 0 \\ c''/\sqrt{1+|c''|^2} & 0 & 1/\sqrt{1+|c''|^2} \end{pmatrix},$$

and get the final form of the submatrix  $N_1$ :

$$N_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now consider the submatrix

$$N_3 = \begin{pmatrix} r & s \\ t & u \\ v & w \end{pmatrix}.$$

If  $v$  and  $w$  are both equal to zero, then  $v_5 \in S_0$ . Therefore, we can assume that  $|v|^2 + |w|^2 \neq 0$  and can apply the transformation  $T = T_1 \oplus T_1 \oplus I \oplus T_1$ , where

$$T_1 = \begin{pmatrix} w/\sqrt{|v|^2+|w|^2} & \bar{v}/\sqrt{|v|^2+|w|^2} \\ -v/\sqrt{|v|^2+|w|^2} & \bar{w}/\sqrt{|v|^2+|w|^2} \end{pmatrix}.$$

Then

$$N_3 = \begin{pmatrix} r' & s' \\ t' & u' \\ 0 & w' \end{pmatrix}, \quad w' = \sqrt{|v|^2 + |w|^2} > 0.$$

If  $s' \neq 0$ , replace  $s'$  by  $|s'|$  by putting  $\tilde{v}_1 = e^{i \arg s'} v_1$ ,  $\tilde{v}_3 = e^{i \arg s'} v_3$ ,  $\tilde{v}_6 = e^{i \arg s'} v_6$ . If  $s' = 0$ , then apply the transformation  $\tilde{v}_1 = e^{-i \arg t'} v_1$ ,  $\tilde{v}_3 = e^{-i \arg t'} v_3$ ,  $\tilde{v}_6 = e^{-i \arg t'} v_6$  and replace  $t'$  by  $|t'|$ . Now we can assume that  $s' \in \Re \geq 0$  and if  $s' = 0$ , then  $t' \in \Re \geq 0$ .

Now let us apply condition (97) of the  $H$ -normality of  $N$ . We obtain:

$$N_3 = \begin{pmatrix} -z_1 \bar{z}_2 \cos \alpha & \sin \alpha \cos \beta \\ z_1 \sin \alpha & z_2 \cos \alpha \cos \beta \\ 0 & \sin \beta \end{pmatrix},$$

$|z_1| = |z_2| = 1$ ,  $0 \leq \alpha, \beta \leq \pi/2$ ,  $\beta \neq 0$ ,  $z_1 = 1$  if  $\sin \alpha \cos \beta = 0$ ,  $z_2 = 1$  if  $\alpha = \pi/2$ . Let us show that in case when  $\alpha = 0$   $N$  is decomposable. Indeed, under the action of (105), where

$$T_2 = \begin{pmatrix} 0 & \bar{p} & (h - \bar{p} z_2 \cos \alpha \cos \beta) / \sin \beta \\ 0 & 0 & 0 \end{pmatrix},$$

the submatrix

$$N_2 = \begin{pmatrix} g & h \\ p & q \end{pmatrix}$$

becomes diagonal. The nondegenerate subspace  $V = \text{span}\{v_1, v_3, v_6\}$  is now invariant for  $N$  and  $N^{[*]}$ , hence,  $N$  is decomposable.

Thus,  $\alpha \neq 0$ . Applying transformation (105) with

$$T_2 = \begin{pmatrix} 0 & t_{14} & t_{15} \\ 0 & t_{24} & t_{25} \end{pmatrix},$$

where

$$\begin{aligned} t_{14} &= g / (z_1 \sin \alpha) \\ t_{15} &= (h - t_{14} z_2 \cos \alpha \cos \beta) / \sin \beta \\ t_{24} &= (p - \bar{t}_{14}) / (z_1 \sin \alpha) \\ t_{25} &= (q - \bar{t}_{24} - t_{24} z_2 \cos \alpha \cos \beta) / \sin \beta, \end{aligned}$$

we reduce  $N_2$  to zero without changing  $N_1$  and  $N_3$ . This is the final form of the matrix  $N - \lambda I$ :

$$N - \lambda I = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -z_1 \bar{z}_2 \cos \alpha & \sin \alpha \cos \beta \\ 0 & 0 & 0 & 0 & 0 & z_1 \sin \alpha & z_2 \cos \alpha \cos \beta \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin \beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$|z_1| = |z_2| = 1, \quad 0 < \alpha, \beta \leq \pi/2, \quad z_1 = 1 \text{ if } \beta = \pi/2, \quad z_2 = 1 \text{ if } \alpha = \pi/2.$$

Show that  $z_1, z_2, \alpha, \beta$  are  $H$ -unitary invariants. Suppose an  $H$ -unitary matrix  $T$  reduces  $N - \lambda I$  to the form

$$\tilde{N} - \lambda I = \begin{pmatrix} 0 & N_1 & 0 \\ 0 & 0 & \tilde{N}_3 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{where}$$

$$N_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \widetilde{N}_3 = \begin{pmatrix} -\widetilde{z}_1\overline{\widetilde{z}_2}\cos\widetilde{\alpha} & \sin\widetilde{\alpha}\cos\widetilde{\beta} \\ \widetilde{z}_1\sin\widetilde{\alpha} & \widetilde{z}_2\cos\widetilde{\alpha}\cos\widetilde{\beta} \\ 0 & \sin\widetilde{\beta} \end{pmatrix},$$

$|\widetilde{z}_1| = |\widetilde{z}_2| = 1$ ,  $0 < \widetilde{\alpha}, \widetilde{\beta} \leq \pi/2$ ,  $\widetilde{z}_1 = 1$  if  $\widetilde{\beta} = \pi/2$ ,  $\widetilde{z}_2 = 1$  if  $\widetilde{\alpha} = \pi/2$ . Therefore,  $T$  has block triangular form (107) and conditions (108) - (114) hold. Combining (108), (114), and (111), we get:  $T_4 = T_1 \oplus t_{55}$  ( $|t_{55}| = 1$ ),  $T_1 = T_6 = T_6^{*-1}$ . Now from (110) it follows that  $T_4 = t_{11} \oplus t_{22}$  ( $|t_{11}| = |t_{22}| = 1$ ),

$$\begin{aligned} t_{22}\sin\alpha\cos\beta &= t_{11}\sin\widetilde{\alpha}\cos\widetilde{\beta} \\ t_{11}z_1\sin\alpha &= t_{22}\widetilde{z}_1\sin\widetilde{\alpha} \\ t_{22}\sin\beta &= t_{55}\sin\widetilde{\beta}, \end{aligned}$$

hence  $t_{11} = t_{22} = t_{55}$ , hence  $N_3 = \widetilde{N}_3$ , i.e.,  $\widetilde{\alpha} = \alpha$ ,  $\widetilde{\beta} = \beta$ ,  $\widetilde{z}_1 = z_1$ ,  $\widetilde{z}_2 = z_2$ . Thus,  $\alpha$ ,  $\beta$ ,  $z_1$ ,  $z_2$  are  $H$ -unitary invariants.

**Lemma 10** *If an indecomposable  $H$ -normal operator  $N$  ( $N : C^7 \rightarrow C^7$ ) has the only eigenvalue  $\lambda$ ,  $\dim S_0 = 2$ , then the pair  $\{N, H\}$  is unitarily similar to canonical pair  $\{(29), (30)\}$ :*

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & -z_1\overline{z_2}\cos\alpha & \sin\alpha\cos\beta \\ 0 & 0 & 0 & \lambda & 0 & z_1\sin\alpha & z_2\cos\alpha\cos\beta \\ 0 & 0 & 0 & 0 & \lambda & 0 & \sin\beta \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$|z_1| = |z_2| = 1, \quad 0 < \alpha, \beta \leq \pi/2, \quad z_1 = 1 \text{ if } \beta = \pi/2, \quad z_2 = 1 \text{ if } \alpha = \pi/2.$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_3 & 0 \\ I_2 & 0 & 0 \end{pmatrix},$$

where  $z_1$ ,  $z_2$ ,  $r$ ,  $\alpha$ ,  $\beta$  are  $H$ -unitary invariants.

**Proof:** We have to prove only the indecomposability of the canonical form because the rest was proved above. The proof, as is customary, is by inductio ad absurdum. Suppose a nondegenerate subspace  $V$  is invariant for  $N$  and  $N^{[*]}$ ; then we can assume (see the proofs of the previous lemmas) that  $\dim V \leq 3$  and  $\exists w_2 = av_6 + bv_7 + v \in V$  ( $v \in (S_0 + S)$ ,  $|a| + |b| \neq 0$ ). Then some nontrivial linear combination of the vectors  $(N^{[*]} - \overline{\lambda}I)w_2 = av_3 + bv_4 + v'$  ( $v' \in S_0$ ) and  $(N - \lambda I)w_2 = a(-z_1\overline{z_2}\cos\alpha v_3 + z_1\sin\alpha v_4) + b(\sin\alpha\cos\beta v_3 + z_2\cos\alpha\cos\beta v_4 + \sin\beta v_5) + v''$  ( $v'' \in S_0$ ) must belong to  $S_0$ . This implies  $b = 0 \Rightarrow a = 0$ . The contradiction obtained proves that  $N$  is indecomposable. The proof is completed.

### 5.3.5 $n = 8$

In this case

$$N - \lambda I = \begin{pmatrix} 0 & N_1 & N_2 \\ 0 & 0 & N_3 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{where } N_1 = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}.$$

As in case when  $n = 7$ , one can check that for the condition  $S \cap S_0 = \{0\}$  to hold the rank of  $N_1$  must be equal to 2. Without loss of generality it can be assumed that

$$\det \begin{pmatrix} a & b \\ e & f \end{pmatrix} \neq 0.$$

As before (in case when  $n = 7$ ), taking the block diagonal transformation  $T = T_1 \oplus I \oplus T_1^{*-1}$ , where

$$T_1 = \begin{pmatrix} a & b \\ e & f \end{pmatrix},$$

we reduce  $N_1$  to the form

$$N_1 = \begin{pmatrix} 1 & 0 & c' & d' \\ 0 & 1 & g' & h' \end{pmatrix}.$$

The results for the previous case  $n = 7$  let reduce the submatrix  $N_1$  to the form  $(I \ 0)$ . Indeed, there exists a transformation

$$T = T_1 \oplus T_2 \oplus T_1^{*-1}, \text{ where } T_2 = T_2^{*-1} = \begin{pmatrix} t_{33} & t_{34} & t_{35} & 0 \\ t_{43} & t_{44} & t_{45} & 0 \\ t_{53} & t_{54} & t_{55} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

that reduces the submatrix  $N_1$  to the form

$$N_1 = \begin{pmatrix} 1 & 0 & 0 & d' \\ 0 & 1 & 0 & h' \end{pmatrix}$$

and there exists a transformation

$$T = T_1 \oplus T_2 \oplus T_1^{*-1}, \text{ where } T_2 = T_2^{*-1} = \begin{pmatrix} t_{33} & t_{34} & 0 & t_{36} \\ t_{43} & t_{44} & 0 & t_{46} \\ 0 & 0 & 1 & 0 \\ t_{63} & t_{64} & 0 & t_{66} \end{pmatrix},$$

that reduces the obtained submatrix  $N_1$  to the desired form

$$N_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{123}$$

Now consider the submatrix  $N_3$  and its submatrices  $N'_3$  and  $N''_3$ :

$$N_3 = \begin{pmatrix} N'_3 \\ N''_3 \end{pmatrix}, \quad N'_3 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad N''_3 = \begin{pmatrix} t & u \\ v & w \end{pmatrix}.$$

Note that  $N''_3$  must be nondegenerate because otherwise the system

$$\begin{aligned} \bar{t}\alpha_1 + \bar{v}\alpha_2 &= 0 \\ \bar{u}\alpha_1 + \bar{w}\alpha_2 &= 0 \end{aligned}$$

has a nontrivial solution  $\{\alpha_i\}_1^2$ , hence, the nonzero vector  $v = \alpha_1 v_5 + \alpha_2 v_6$  belongs to  $S_0$ .

Thus,  $N''_3$  is nondegenerate. Recall that any nondegenerate matrix is a product of some selfadjoint positive definite matrix and some unitary one. Consequently,  $N''_3 = RU$ , where  $R$  is selfadjoint positive definite and  $U$  is unitary. Let  $U_1$  be a unitary matrix reducing  $R$  to the real positive diagonal form. Taking  $T = U^*U_1 \oplus U^*U_1 \oplus U_1 \oplus U^*U_1$ , we carry  $N''_3$  into the form

$$N''_3 = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \quad r_1, r_2 \in \Re, \quad 0 < r_1 \leq r_2$$

without changing the submatrix  $N_1$ . Now we have

$$N_3 = \begin{pmatrix} N'_3 \\ N''_3 \end{pmatrix} = \begin{pmatrix} p' & q' \\ r' & s' \\ r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

Further, apply transformation (105) with

$$T_2 = \begin{pmatrix} 0 & \bar{m} & (k - r'\bar{m})/r_1 & (l - s'\bar{m})/r_2 \\ 0 & 0 & 0 & n/r_2 \end{pmatrix}$$

and reduce the submatrix

$$N_2 = \begin{pmatrix} k & l \\ m & n \end{pmatrix}$$

to zero. Finally apply condition (97) of the  $H$ -normality of  $N$ . We get:  $r_2 \leq 1$ . Show that if  $r_1 = r_2$ , then  $N$  is decomposable. In fact, if  $r_1 = r_2 = 1$ , then from (97) it follows that  $N'_3 = 0$ , hence, the nondegenerate subspace  $V = \text{span}\{v_1, v_3, v_5, v_7\}$  is invariant for  $N$  and  $N^{[*]}$ , hence,  $N$  is decomposable. If  $r_1 = r_2 < 1$ , then the matrix  $N'_3/\sqrt{1-r_1^2}$  is unitary, therefore, there exists a unitary matrix  $U$  that reduces  $N'_3$  to the diagonal form. Then the transformation  $T = U \oplus U \oplus U \oplus U$  does not change the submatrices  $N_1 = (I \ 0)$ ,  $N_2 = 0$ ,  $N''_3 = r_1 I$  and reduces  $N'_3$  to the diagonal form. Now it is seen that  $N$  is decomposable ( $V = \text{span}\{v_1, v_3, v_5, v_7\}$  is nondegenerate,  $NV \subseteq V$ ,  $N^{[*]}V \subseteq V$ ). Thus, in either case  $N$  is decomposable.

There remains to consider the case when  $r_1 < r_2$ . If  $q' \neq 0$ , let us replace  $q'$  by  $|q'|$  by means of the transformation  $\tilde{v}_1 = e^{i \arg q'} v_1$ ,  $\tilde{v}_3 = e^{i \arg q'} v_3$ ,  $\tilde{v}_5 = e^{i \arg q'} v_5$ ,  $\tilde{v}_7 = e^{i \arg q'} v_7$ . If  $q' = 0$ , let us put  $\tilde{v}_1 = e^{-i \arg r'} v_1$ ,  $\tilde{v}_3 = e^{-i \arg r'} v_3$ ,  $\tilde{v}_5 = e^{-i \arg r'} v_5$ ,  $\tilde{v}_7 = e^{-i \arg r'} v_7$ . Then  $r'$  will be replaced by  $|r'|$ . Thus, one can assume that  $q' \in \Re \geq 0$  and if  $q' = 0$ , then  $r' \in \Re \geq 0$ . Applying (97) and renaming the terms of  $N_3$ , we get

$$N_3 = \begin{pmatrix} -z_1 \bar{z}_2 \sin \alpha \cos \beta & \cos \alpha \cos \gamma \\ z_1 \cos \alpha \cos \beta & z_2 \sin \alpha \cos \gamma \\ \sin \beta & 0 \\ 0 & \sin \gamma \end{pmatrix}, \quad (124)$$

$|z_1| = |z_2| = 1$ ,  $0 < \beta < \gamma \leq \pi/2$ ,  $0 \leq \alpha \leq \pi/2$ ,  $z_1 = 1$  if  $\cos \alpha \cos \gamma = 0$ ,  $z_2 = 1$  if  $\alpha = 0$ . We already know that if  $N'_3$  is diagonal,  $N$  is decomposable. Therefore,  $\alpha \neq \pi/2$ . As a result, we have:

$$N - \lambda I = \begin{pmatrix} 0 & N_1 & 0 \\ 0 & 0 & N_3 \\ 0 & 0 & 0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (125)$$

$N_3$  has form (124),

$$\begin{aligned} |z_1| = |z_2| = 1, \quad 0 < \beta < \gamma \leq \pi/2, \quad 0 \leq \alpha < \pi/2, \\ z_1 = 1 \text{ if } \gamma = \pi/2, \quad z_2 = 1 \text{ if } \alpha = 0. \end{aligned} \quad (126)$$

Check the  $H$ -unitary invariance of the numbers  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $z_1$ , and  $z_2$ . To this end suppose that an  $H$ -unitary matrix  $T$  reduces  $N - \lambda I$  to the form  $\tilde{N} - \lambda I$ , where  $N - \lambda I$  has form (125), (124), (126),

$$\tilde{N} - \lambda I = \begin{pmatrix} 0 & N_1 & 0 \\ 0 & 0 & \tilde{N}_3 \\ 0 & 0 & 0 \end{pmatrix},$$

$N_1$  has form (123),  $N_3$  has form (124),

$$\tilde{N}_3 = \begin{pmatrix} -\tilde{z}_1 \bar{\tilde{z}}_2 \sin \tilde{\alpha} \cos \tilde{\beta} & \cos \tilde{\alpha} \cos \tilde{\gamma} \\ \tilde{z}_1 \cos \tilde{\alpha} \cos \tilde{\beta} & \tilde{z}_2 \sin \tilde{\alpha} \cos \tilde{\gamma} \\ \sin \tilde{\beta} & 0 \\ 0 & \sin \tilde{\gamma} \end{pmatrix},$$

$$\begin{aligned} |z_1| = |z_2| = 1, \quad 0 < \tilde{\beta} < \tilde{\gamma} \leq \pi/2, \quad 0 \leq \tilde{\alpha} < \pi/2, \\ \tilde{z}_1 = 1 \text{ if } \tilde{\gamma} = \pi/2, \quad \tilde{z}_2 = 1 \text{ if } \tilde{\alpha} = 0. \end{aligned}$$

Then  $T$  has form (107) and conditions (108) - (114) hold. From (108), (114), and (111) it follows that  $T_4 = T_1 \oplus T'_4$ ,  $T'_4 T_4^* = I$ ,  $T_1 = T_6 = T_6^{*-1}$ . From (110) it follows that  $N''_3 T_1 = T'_4 \tilde{N}_3$ . Taking into account general form (122) of a  $2 \times 2$  unitary matrix, we can check that this equality implies  $T'_4 = T_1 = t_{11} \oplus t_{22}$  ( $|t_{11}| = |t_{22}| = 1$ ),  $\tilde{\beta} = \beta$ ,  $\tilde{\gamma} = \gamma$ . Applying (110) again, we get

$$\begin{aligned} t_{22} \cos \alpha \cos \gamma &= t_{11} \cos \tilde{\alpha} \cos \tilde{\gamma} \\ t_{11} z_1 \cos \alpha \cos \beta &= t_{22} \tilde{z}_1 \cos \tilde{\alpha} \cos \tilde{\beta}, \end{aligned}$$

hence  $t_{11} = t_{22}$ , hence  $\tilde{N}_3 = N_3$ , i.e.,  $\tilde{\alpha} = \alpha$ ,  $\tilde{z}_1 = z_1$ ,  $\tilde{z}_2 = z_2$ .

**Lemma 11** *If an indecomposable  $H$ -normal operator  $N$  ( $N : C^8 \rightarrow C^8$ ) has the only eigenvalue  $\lambda$ ,  $\dim S_0 = 2$ , then the pair  $\{N, H\}$  is unitarily similar to canonical pair  $\{(31), (32)\}$ :*

$$N - \lambda I = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -z_1 \bar{z}_2 \sin \alpha \cos \beta & \cos \alpha \cos \gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & z_1 \cos \alpha \cos \beta & z_2 \sin \alpha \cos \gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin \gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$|z_1| = |z_2| = 1, \quad 0 \leq \alpha < \pi/2, \quad 0 < \beta < \gamma \leq \pi/2, \\ z_1 = 1 \text{ if } \gamma = \pi/2, \quad z_2 = 0 \text{ if } \alpha = 0.$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_4 & 0 \\ I_2 & 0 & 0 \end{pmatrix},$$

where  $z_1, z_2, \alpha, \beta, \gamma$  are  $H$ -unitary invariants.

**Proof:** We must prove only the indecomposability of the canonical form. Assume the converse. Then (see the proofs of the previous lemmas) we can assume that  $\dim V \geq 4$ ,  $w_2 = av_7 + bv_8 + v \in V$  ( $v \in (S_0 + S)$ ,  $|a| + |b| \neq 0$ ). The vectors  $(N - \lambda I)(N^{[*]} - \bar{\lambda}I)w_2 = av_1 + bv_2$ ,  $(N^{[*]} - \bar{\lambda}I)^2 w_2 = a(-\bar{z}_1 z_2 \sin \alpha \cos \beta v_1 + \cos \alpha \cos \gamma v_2) + b(\bar{z}_1 \cos \alpha \cos \beta v_1 + \bar{z}_2 \sin \alpha \cos \gamma v_2)$  and  $(N - \lambda I)^2 w_2 = a(-z_1 \bar{z}_2 \sin \alpha \cos \beta v_1 + z_1 \cos \alpha \cos \beta v_2) + b(\cos \alpha \cos \gamma v_1 + z_2 \sin \alpha \cos \gamma v_2)$  must be collinear because otherwise we get  $S_0 \subset V$ , but since the condition  $NS_1 \subset (S_1 + S_0)$  does not hold, we obtain  $\dim V > 4$ . Thus, let us write the conditions of the linear dependence (if  $a$  or  $b$  is equal to zero, the vectors are not collinear):

$$-\bar{z}_1 z_2 \sin \alpha \cos \beta + \bar{z}_1 \cos \alpha \cos \beta \frac{b}{a} = \cos \alpha \cos \gamma \frac{a}{b} + \bar{z}_2 \sin \alpha \cos \gamma \\ -z_1 \bar{z}_2 \sin \alpha \cos \beta + \cos \alpha \cos \gamma \frac{b}{a} = z_1 \cos \alpha \cos \beta \frac{a}{b} + z_2 \sin \alpha \cos \gamma.$$

If we replace the last condition by its complex conjugate and subtract it from the first, we obtain:

$$\bar{z}_1 \cos \alpha \cos \beta \frac{b}{a} - \cos \alpha \cos \gamma \left(\frac{\bar{b}}{a}\right) = \cos \alpha \cos \gamma \frac{a}{b} - \bar{z}_1 \cos \alpha \cos \beta \left(\frac{\bar{a}}{b}\right)$$

or

$$\bar{z}_1 \cos \alpha \cos \beta \frac{|a|^2 + |b|^2}{a\bar{b}} = \cos \alpha \cos \gamma \frac{|a|^2 + |b|^2}{ab}.$$

Modulus of the left hand side must be equal to that of the right hand side, i.e.,  $\cos \alpha \cos \beta = \cos \alpha \cos \gamma$ . Since  $\cos \alpha \neq 0$ ,  $\cos \beta = \cos \gamma$ , hence,  $\beta = \gamma$ . But for our canonical form  $\beta < \gamma$ . This contradiction proves the indecomposability of the operator  $N$ .

We have considered all alternatives for an indecomposable operator  $N$  and have obtained canonical forms for each case. Thus, we have proved Theorem 2.

## Appendix

### Canonical Forms for $2 \times 2$ Matrices under Congruence

**Proposition 3** *Any invertible matrix  $A$  of order  $2 \times 2$  is congruent to one and only one of the following canonical forms:*

$$A = \begin{pmatrix} z & \varrho e^{-i\pi/3} z \\ 0 & e^{i\pi/3} z \end{pmatrix}, \quad |z| = 1, \quad \varrho \in \Re \geq \sqrt{3}, \quad 0 \leq \arg z < \pi \text{ if } \varrho > \sqrt{3}, \quad (127)$$

$$A = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}, \quad |z_1| = 1, |z_2| = 1, \arg z_1 \leq \arg z_2, \quad (128)$$

where  $z, z_1, z_2, \varrho$  form a complete a minimal set of invariants.

**Proof:** Consider the matrix  $A' = AA^*{}^{-1}$ . If  $\tilde{A} = TAT^*$ , then  $\tilde{A}' = TA'T^{-1}$  so that spectral properties of  $A'$  do not change under congruence of  $A$ . Reduce  $A'$  to the Jordan normal form. Since  $|\det A'| = 1$ , there exist three such forms:

$$A' = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \quad x_1 \neq x_2, |x_1 x_2| = 1, |x_1| \leq 1, \quad (129)$$

$$A' = xI, \quad |x| = 1, \quad (130)$$

$$A' = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, \quad |x| = 1. \quad (131)$$

(a)  $A'$  is reduced to form (129). Since  $A = A'A^*$ , we have

$$A' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a}x_1 & \bar{c}x_1 \\ \bar{b}x_2 & \bar{d}x_2 \end{pmatrix} = A'A^*. \quad (132)$$

It is seen that either  $b = c = 0$  or  $\arg x_1 = \arg x_2$ .

If  $|x_1| < 1$ , then from (132) it follows that  $a = d = 0$ ; since  $A$  is invertible,  $b$  and  $c$  are nonzero, therefore,  $\arg x_1 = \arg x_2$ . Now let us consider the function  $f(\varrho) = \frac{1}{2}(1 - \varrho^2 - \sqrt{(\varrho^2 + 1)(\varrho^2 - 3)})$  of the real variable  $\varrho$ . It monotonically decreases on the interval  $(\sqrt{3}, +\infty)$ ,  $f(\sqrt{3}) = -1$ , and  $\lim_{\varrho \rightarrow +\infty} f(\varrho) = -\infty$ , therefore, the equation  $f(\varrho) = s$  has a root  $\varrho > \sqrt{3}$  for all  $s < -1$ . Let  $\varrho$  be a root of the equation  $f(\varrho) = -|x_2|$  and let  $e^{i \arg x_2} = -e^{i\pi/3} z^2$ , where  $|z| = 1, 0 \leq \arg z < \pi$ . Then  $x_1 = \frac{1}{2}e^{i\pi/3} z^2(1 - \varrho^2 + \sqrt{(\varrho^2 + 1)(\varrho^2 - 3)})$ ,  $x_2 = \frac{1}{2}e^{i\pi/3} z^2(1 - \varrho^2 - \sqrt{(\varrho^2 + 1)(\varrho^2 - 3)})$ , and from (132) it follows that

$$A = \begin{pmatrix} 0 & b \\ e^{i\pi/3} z^2 f(\varrho) \bar{b} & 0 \end{pmatrix}, \quad b \neq 0.$$

Now the transformation

$$T = \begin{pmatrix} 1 & \bar{z}(e^{-i\pi/3} f(\varrho) - 1)/(\bar{b}(f(\varrho)^2 - 1)) \\ e^{2i\pi/3} \varrho f(\varrho)/(e^{i\pi/3} f(\varrho) - 1) & -e^{i\pi/3} \bar{z} \varrho/(\bar{b}(f(\varrho)^2 - 1)) \end{pmatrix}$$

reduces  $A$  to form (127) with  $\varrho > \sqrt{3}$ . The numbers  $\varrho$  and  $z$  cannot be changed under congruence because the eigenvalues of  $A'$  are invariants and from the condition  $e^{i\pi/3} z^2 f(\varrho) = e^{i\pi/3} \tilde{z}^2 f(\tilde{\varrho})$  ( $|z| = |\tilde{z}| = 1, 0 \leq \arg z, \arg \tilde{z} < \pi, \varrho, \tilde{\varrho} \in \Re > \sqrt{3}$ ) it follows that  $\tilde{z} = z, \tilde{\varrho} = \varrho$ .

If  $|x_1| = 1$ , then from the condition  $x_1 \neq x_2$  it follows that  $\arg x_1 \neq \arg x_2$ , hence  $b = c = 0$ . By taking  $T = D_2$  one can interchange the terms  $a$  and  $d$  of the matrix  $A$ . Hence, we can assume that  $\arg a \leq \arg d$ . Applying the transformation

$$T = \begin{pmatrix} 1/\sqrt{|a|} & 0 \\ 0 & 1/\sqrt{|d|} \end{pmatrix},$$

we reduce  $A$  to form (128) with  $z_1 = e^{i \arg a}, z_2 = e^{i \arg d}$ .

To prove the invariance of  $z_1$  and  $z_2$  suppose that  $\tilde{A} = TAT^*$ , where  $A = z_1 \oplus z_2, \tilde{A} = \tilde{z}_1 \oplus \tilde{z}_2, |z_1| = |z_2| = |\tilde{z}_1| = |\tilde{z}_2| = 1, \arg z_1 \leq \arg z_2, \arg \tilde{z}_1 \leq \arg \tilde{z}_2$ . Then

$$z_1 |t_{11}|^2 + z_2 |t_{12}|^2 = \tilde{z}_1 \quad (133)$$

$$z_1 t_{11} \bar{t}_{21} + z_2 t_{12} \bar{t}_{22} = 0 \quad (134)$$

$$z_1 \bar{t}_{11} t_{21} + z_2 \bar{t}_{12} t_{22} = 0 \quad (135)$$

$$z_1 |t_{21}|^2 + z_2 |t_{22}|^2 = \tilde{z}_2. \quad (136)$$

Since  $t_{11} \bar{t}_{21} = -\bar{z}_1 z_2 t_{12} \bar{t}_{22}$  (condition (134)), (135) holds only if  $(z_2^2 - z_1^2) \bar{t}_{12} t_{22} = 0$ . If  $z_1^2 \neq z_2^2$ , then  $t_{12}$  must be zero because if  $t_{22} = 0$ , then  $t_{11} = 0$  and, therefore,  $\tilde{z}_1 = z_2, \tilde{z}_2 = z_1$ , which contradicts the condition

$\arg \tilde{z}_1 \leq \arg \tilde{z}_2$ . Thus,  $t_{12} = 0$ , hence,  $t_{21} = 0$ ,  $\tilde{z}_1 = z_1$ ,  $\tilde{z}_2 = z_2$ . If  $z_1 = z_2$ , then, according to (133) - (136),  $\tilde{z}_1 = z_1(|t_{11}|^2 + |t_{12}|^2)$ ,  $\tilde{z}_2 = z_1(|t_{21}|^2 + |t_{22}|^2)$ , hence  $\tilde{z}_1 = \tilde{z}_2 = z_1 = z_2$ . If  $z_2 = -z_1$  and  $\overline{t_{12}}t_{22} \neq 0$ , then  $t_{11}\overline{t_{21}} \neq 0$  and  $\tilde{z}_1 = z_1(|t_{11}|^2 - |t_{12}|^2)$ . Since  $|t_{21}|/|t_{22}| = |t_{12}|/|t_{11}|$ ,  $\tilde{z}_2 = z_1(|t_{21}|^2 - |t_{22}|^2) = -\tilde{z}_1|t_{22}|^2/|t_{11}|^2$ . As  $\arg \tilde{z}_1 \leq \arg \tilde{z}_2$ , we get  $\tilde{z}_1 = z_1$ ,  $\tilde{z}_2 = z_2$ . The case when  $z_2 = -z_1$  and  $\overline{t_{12}}t_{22} = 0$  can be considered as before. Thus, we have proved the invariance of the numbers  $z_1$  and  $z_2$ .

(b)  $A'$  is reduced to form (130). Then  $A = xA^*$ ,  $|x| = 1$ , this property being invariant with respect to congruence. Since  $A$  is invertible,  $A = RU$ , where  $R$  is selfadjoint positive definite matrix and  $U$  is unitary. Let  $T$  be a unitary matrix reducing  $U$  to the diagonal form  $\Lambda$ . After the application of  $T$  we have:  $A = \tilde{R}\Lambda$ , where  $\tilde{R} = TRT^*$  is also selfadjoint positive definite. Now let  $T$  be a lowertriangular matrix such that  $T\tilde{R}T^* = I$ . Then we reduce  $A$  to the uppertriangular form  $T^{*-1}\Lambda T^*$ . Since the term  $c$  of  $A$  is now equal to zero, from the condition  $A = xA^*$  it follows that  $b$  is also equal to zero, i.e.,  $A$  is diagonal. We already know that a diagonal matrix is congruent to (128) (see case (a) above). Thus,  $A$  can be reduced to form (128).

(c)  $A'$  is reduced to form (131). Let  $x = -e^{i\pi/3}z^2$  ( $|z| = 1$ ). Then the application of the condition  $A = A'A^*$  yields:

$$A = \begin{pmatrix} a & b \\ -e^{i\pi/3}z^2\bar{b} & 0 \end{pmatrix}, \quad b = \bar{a} + e^{-i\pi/3}\bar{z}^2a.$$

For  $A$  to be invertible  $b$  must be nonzero. Since  $|b| = |a + e^{i\pi/3}z^2\bar{a}| = |a\bar{z} + e^{i\pi/3}\bar{a}z| = |a\bar{z} - e^{-2i\pi/3}\bar{a}z| = |e^{i\pi/3}a\bar{z} - e^{-i\pi/3}\bar{a}z| = 2|\Im\{e^{i\pi/3}a\bar{z}\}|$ , we see that  $\Im\{e^{i\pi/3}a\bar{z}\} \neq 0$ . Let us chose  $z$  so that  $\Im\{e^{i\pi/3}a\bar{z}\} > 0$ . Applying the transformation

$$T = \frac{\sqrt[4]{3}}{\sqrt{|b|}^3} \begin{pmatrix} |b| & \frac{2}{3}i\bar{z}\Im\{a\bar{z}\}|b|/\bar{b} \\ e^{i\pi/3}\bar{z}\bar{b} & \bar{z}^2(-\frac{2}{3}i\Im\{a\bar{z}\} + a\bar{z}) \end{pmatrix},$$

we reduce  $A$  to form (128) with  $\varrho = \sqrt{3}$ . It is clear that matrix (128) with  $\varrho = \sqrt{3}$  is not congruent to that with  $\varrho > 3$  because in the former case  $A'$  has the diagonal Jordan normal form in contrast to the latter. Therefore, we must prove only the invariance of  $z$ . Note that if  $\tilde{A} = TAT^*$ , where

$$A = \begin{pmatrix} z & \sqrt{3}e^{-i\pi/3}z \\ 0 & e^{i\pi/3}z \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} \tilde{z} & \sqrt{3}e^{-i\pi/3}\tilde{z} \\ 0 & e^{i\pi/3}\tilde{z} \end{pmatrix}, \quad |z| = |\tilde{z}| = 1,$$

then  $\tilde{z}^2 = z^2$  because the eigenvalue  $x = -e^{i\pi/3}z^2$  of  $A'$  does not change under congruence of  $A$ . Therefore,

$$A' = z^2 \begin{pmatrix} 1 - 3e^{i\pi/3} & \sqrt{3} \\ \sqrt{3} & e^{2i\pi/3} \end{pmatrix} = \tilde{A}'.$$

For  $T$  to satisfy the condition  $A'T = TA'$  the matrix  $T$  must has the form

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{11} + it_{12} \end{pmatrix}.$$

Now from the condition  $\tilde{A} = TAT^*$  it follows that

$$z|t_{11}|^2 + \sqrt{3}e^{-i\pi/3}zt_{11}\overline{t_{12}} + e^{i\pi/3}z|t_{12}|^2 = \tilde{z} \quad (137)$$

$$z\overline{t_{11}}t_{12} + \sqrt{3}e^{-i\pi/3}z|t_{12}|^2 + e^{i\pi/3}z(t_{11}\overline{t_{12}} + i|t_{12}|^2) = 0. \quad (138)$$

If  $t_{12} \neq 0$ , from (138) it follows that

$$e^{-i\pi/6}\frac{\overline{t_{11}}}{t_{12}} + \sqrt{3}e^{-i\pi/2} + e^{i\pi/6}\frac{t_{11}}{t_{12}} + e^{2i\pi/3} = 0,$$

which is impossible because the imaginary part of the left hand side is equal to  $\Im\{\sqrt{3}e^{-i\pi/2} + e^{2i\pi/3}\} = -\sqrt{3}/2$ . Therefore,  $t_{12} = 0$ , hence (condition (137))  $\tilde{z} = z$ , i.e.,  $z$  is an invariant. This concludes the proof of the proposition.

## References

- [1] I. Gohberg, B. Reichstein, *On classification of Normal Matrices in an Indefinite Scalar Product*, Integral Equations and Operator Theory, 13 (1990), 364-394.