New coins from old, smoothly

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Abstract

Given a (known) function $f : [0,1] \to (0,1)$, we consider the problem of simulating a coin with probability of heads $f(p)$ by tossing N times a coin with unknown heads probability p, where N may be random. Keane and O'Brien (1994) showed that such a simulation scheme with $\mathbb{P}_p(N < \infty) = 1$ exists iff f is continuous. Nacu and Peres (2005) proved that f is real analytic iff such a simulation scheme exists with $\mathbb{P}_p(N > n)$ decaying exponentially. We prove that for $\alpha > 0$ noninteger, f is in the space $C^\alpha[0,1]$ if and only if a simulation scheme as above exists with $\mathbb{P}_p(N > n) \leq C\Delta_n(p)$,where $\Delta_n(x) := \max\{\sqrt{x(1-x)/n}, 1/n\}$. The key to the proof is a new result in approximation theory: Let H_n^+ be the space of homogenous polynomials of degree n with non-negative coefficients in two variables. We show that a function $f : [0,1] \to (0,1)$ is in $C^\alpha[0,1]$ if and only if f has a series representation $\sum_{n=1}^\infty F_n(x, 1-x)$ with $F_n(x,y) \in H_n^+$ and $\sum_{k>n} F_k(x, 1-x) \leq C\Delta_n(x)$ for all $x \in [0,1]$ and $n \geq 1$. We also provide a counterexample to a theorem stated without proof by Lorentz (1963), who claimed that if some $\varphi_n(x,y) \in H_n^+$ satisfy $|f(x) - \varphi_n(x, 1-x)| \leq C\Delta_n(x)$ for all $x \in [0,1]$ and $n \geq 1$, then $f \in C^\alpha[0,1]$.

1 Introduction

Given a coin with unknown probability of heads $p \in [0,1]$, we would like to simulate a coin with probability of heads $f(p)$ where $f : [0,1] \to (0,1)$ is a known function. This means that we are allowed to toss the original p-coin N times, where N is an almost surely finite stopping time, and declare heads or tails, depending on the outcome of the first N tosses. The probability of declaring a head must be exactly $f(p)$.

This problem goes back to von Neumann's article [9] where he showed how to simulate a fair coin (i.e., $f(p) = 1/2$) using a biased p-coin. The question was subsequently answered for various other classes of functions – see [2, 7, 5, 6]. In particular, it was shown in [2] that f is simulable

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for all p in a closed interval $D \subseteq [0,1]$ if and only if f is continuous in D. (A further condition is needed if f can take the values $0,1$, which we have excluded.) In [5], it was shown that f can be simulated by a finite automaton for all $p \in (0,1)$ if an only if f is a rational function over $\mathbb{Q}$. In [6], a simulation with exponential tails was shown to be possible for p in a closed interval D if and only if f is real analytic in D. Moreover, the problem of simulation was restated in [6] as an approximation problem, and the question of characterizing simulation rates for non-analytic functions was posed.

Definition 1. *Given a simulation algorithm, its* ***simulation rate*** *is the probability* $\mathbb{P}_p(N > n)$ *that the number of required inputs exceeds* n. *If a simulation algorithm with* $\mathbb{P}_p(N > n) = O(\psi_n(p))$ *exists, we say that the function is* ***simulable at the rate*** $\psi_n(p)$.

The goal of this paper is to show that the simulation rate is determined by the smoothness of the simulated function f. We prove that for $\alpha \notin \mathbb{Z}$, a function $f : [0,1] \to (0,1)$ is in the space C^α (defined by a Hölder condition on the derivative of order $\lfloor \alpha \rfloor$) if and only if f can be simulated at the rate $O(n^{-\alpha/2})$ away from the boundary points $0,1$ and at a faster rate near the boundary– see Theorem 9 below for the precise statement.

2 Preliminaries and statement of results

We first recall relevant definitions and results from the literature on this problem and from approximation theory. Recall that the univariate ***Bernstein polynomials*** (see, e.g., [4]) are defined as

$$x \mapsto p_{nk}(x) := \binom{n}{k} x^k (1-x)^{n-k}, \qquad k = 0, \dots, n. \tag{1}$$

Each such univariate polynomial has its bivariate counterpart

$$(x,y) \mapsto \widetilde{p}_{nk}(x,y) := \binom{n}{k} x^k y^{n-k}, \qquad k = 0, \dots, n.$$

This rule extends to a correspondence between polynomials in ***Bernstein form*** and their bivariate counterparts:

$$p(x) := \sum_{k=0}^{n} a_k p_{nk}(x) \mapsto \widetilde{p}(x,y) := \sum_{k=0}^{n} a_k \widetilde{p}_{nk}(x,y).$$

Throughout this paper, the symbol $\widetilde{p}$ will be reserved for the bivariate version of a univariate polynomial p in Bernstein form. Note that the multiplication by $1 = x + (1-x)$ does not change the value of $p(x)$ but it does change the Bernstein form of p and therefore its bivariate version $\widetilde{p}$. The operation$\widetilde{\ }$ will be therefore applied only to polynomials explicitly given in Bernstein form.

To compare bivariate polynomials of the same degree (and therefore also polynomials in Bernstein form of the same degree), we will use the standard partial order:

Definition 2. *Given two homogeneous polynomials* q *and* r *of equal degree with real coeffcients, we write* $q \preceq r$ *to denote that the coefficients of the polynomial* $r - q$ *are nonnegative.*

Result 3 below was established in [6] using techniques from probability. This result provides us with a reduction of the original problem to a problem in approximation theory, which we address in this paper.

Result 3 ([6]). *If there exists an algorithm that simulates a function f on a set $D \subset [0,1]$, then for all $n \geq 1$ there exist univariate polynomials*

$$g_n(x) := \sum_{k=0}^{n} \binom{n}{k} a(n,k) x^k (1-x)^{n-k}, \qquad h_n(x) := \sum_{k=0}^{n} \binom{n}{k} b(n,k) x^k (1-x)^{n-k}$$

with the following properties:

(i) $0 \leq a(n,k) \leq b(n,k) \leq 1$.

(ii) $\binom{n}{k} a(n,k)$, $\binom{n}{k} b(n,k)$ *are integers.*

(iii) $\lim_{n\to\infty} g_n(p) = f(p) = \lim_{n\to\infty} h_n(p)$ *for all* $p \in D$.

(iv) *For all* $m < n$, $(x+y)^{n-m}\widetilde{g}_m(x,y) \preceq \widetilde{g}_n(x,y)$ *and* $(x+y)^{n-m}\widetilde{h}_m(x,y) \succeq \widetilde{h}_n(x,y)$.

Conversely, if there exist such polynomials g_n, f_n satisfying (i) – (iv), then there exists an algorithm that simulates f on D at the rate

$$\mathbb{P}_p(N > n) = h_n(p) - g_n(p).$$

The requirement (ii) is in fact extraneous, as was already shown in [6], since one can always round the values $\binom{n}{k} a(n,k)$ down and the values $\binom{n}{k} b(n,k)$ up to an integer if needed.

Thus the problem of determining the rate of simulation is equivalent to the problem of determining the order of two-sided approximation to f, from above by polynomials g_n and from below by polynomials h_n, both of degree at most n, and satisfying requirements (i) and (iv). We will refer to requirements (iv) as the ***consistency requirements,*** to the approximation scheme (g_n) as a ***Bernstein-positive consistent approximation from below,*** and to the approximation scheme (h_n) as a ***Bernstein-positive consistent approximation from above.***

Observe that a Bernstein-positive consistent approximation to a function f from below is equivalent to a certain nonnegative series representation of f. Here is a precise statement.

Lemma 4. *Let (r_n) be a (weakly) decreasing sequence converging to 0 and let I be any subinterval of $[0,1]$. A function f is approximable on I by a sequence of nonnegative Bernstein polynomials (g_n) of degree n satisfying the consistency requirement (iv)*

$$(x+y)^{n-m}\widetilde{g}_m(x,y) \preceq \widetilde{g}_n(x,y) \quad \text{for all } \ n \geq m \tag{2}$$

and for all $x \in I$,

$$0 \leq f(x) - g_n(x) \leq r_n \tag{3}$$

if and only if f can be represented as a series

$$f(x) = \sum_{n=0}^{\infty} f_n(x) \quad \text{where} \quad \sum_{n>N} f_n(x) \leq r_N \quad \text{for all } \ x \in I, \tag{4}$$

where each f_n is a polynomial in Bernstein form of degree n with nonnegative coefficients.

Proof. Given an approximation scheme (g_n) as above, set $f_n(x) := g_n(x) - g_{n-1}(x)$ where the second term $g_{n-1}(x)$ is rewritten in Bernstein form of degree n and where $g_0(x) := 0$. The consistency requirement (2) then guarantees that the Bernstein coefficients of f_n are nonnegative, and the sum $\sum_{n>N} f_n(x)$ telescopes into $f(x) - g_N(x)$, which is bounded by r_N according to (3).

Conversely, given a series representation (4), let $g_n(x) := \sum_{k \le n} f_k(x)$. Then the difference $f_n(x) := g_n(x) - g_{n-1}(x)$ a Bernstein polynomial with nonnegative coefficients, which means exactly that the polynomials (g_n) satisfy the consistency requirement (2). Also, $f(x) - g_n(x) = \sum_{k>n} f_k(x) \le r_n$ due to the rate condition (3). □

This approximation problem can be contrasted with the classical approximation of a given function by (unrestricted) polynomials of degree at most n on the interval $[0,1]$. In that case, the approximation order coincides with the smoothness of f. To state this classical result precisely, we first recall how smoothness is measured.

Definition 5. *Let $\alpha > 0$. A function f is said to be in the* ***smoothness class*** $C^\alpha[0,1]$ *if f is $r := \lceil \alpha \rceil - 1$ times differentiable and the following condition holds:*

The ***modulus of continuity*** *of the rth derivative $f^{(r)}$*

$$\omega(f^{(r)}; h) := \sup_{t<h,\ x \in [0,1-t]} |f^{(r)}(x+t) - f^{(r)}(x)|$$

is of order $O(h^{\alpha - r})$. In that case, we will use the notation

$$\|f\|_{C^\alpha} := \sup_h \frac{\omega(f^{(r)}; h)}{h^{\alpha - r}}.$$

The order of approximating a given function by polynomials is then determined as follows.

Result 6 (see, e.g., [1, Chapter 8, Theorem 6.3]). *Let $\alpha > 0$ be a non-integer. The best approximant p_n to f from the space Π_n of polynomials of degree at most n satisfies*

$$|p_n(x) - f(x)| = O(\Delta_n^{2\alpha}(x)) \qquad \text{for all } x \in [0,1]$$

if and only if $f \in C^\alpha[0,1]$. Here the quantity $\Delta_n(x)$ is defined by

$$\Delta_n := \Delta_n(x) := \max\left\{ \sqrt{\frac{x(1-x)}{n}}, \frac{1}{n} \right\}.$$

In other words, the rate of approximation of $f \in C^\alpha[0,1]$ is $O(n^{-\alpha})$ inside the interval $[0,1]$ and is $O(n^{-2\alpha})$ close to the endpoints 0 and 1. The characterization of best polynomial approximation for integer values α involves the so-called generalized Lipschitz class, which will be described in Section 7. In the main part of this paper, we work under the assumption $\alpha \notin \mathbb{N}$.

Result 6 shows that the best simulation rate for a function $f \in C^\alpha[0,1]$ cannot exceed $O(\Delta_n^{2\alpha})$. However, since our approximants must satisfy special restrictions imposed by Result 3, we should not expect to achieve the approximation order provided by unrestricted polynomials Π_n.

In view of requirement (i), it is natural to consider first the approximation order achieved by polynomials with nonnegative Bernstein coefficients. G. G. Lorentz proposed a solution to this problem in [3], where he argued that thus restricted approximation order is half the approximation order provided by unrestricted polynomials, i.e., half the smoothness of the function f. Theorem 1 of [3] establishes that a C^α-function f can be approximated at the rate $O(\Delta_n^\alpha)$ by nonnegative Bernstein polynomials of degree n.

Result 7 ([3, Theorem 1]). *Let* $\alpha > 0$*. A function* $f \in C^{\alpha}[0,1]$ *can be approximated by polynomials* q_n *of degree at most* n *with nonnegative Bernstein coefficients at the rate*

$$|q_n(x) - f(x)| = O(\Delta_n^{\alpha}(x)) \qquad \text{for all } x \in [0,1]. \tag{5}$$

Lorentz [3] also stated (without proof) a converse to this result; unfortunately that converse is incorrect. We return to this point at the end of the section.

We will use a variant of Lorentz' approach to demonstrate that, with all the extra requirements (i) – (iv) in place, we can still achieve the same approximation order as in (5). This means that the rate of simulation is half the smoothness of the simulated function f.

We begin by proving a reduction lemma that shows that it is enough to find consistent approximants g_n, h_n for each b-adic degree $n = b^l$ where b is a fixed integer greater than 1. Using these approximants, one can then interpolate between b-adic levels to build up a consistent approximation scheme providing the same approximation order as the dyadic polynomials g_n, h_n, $n = b^l$. This b-adic idea *per se* is quite well known and, in particular, is used in [6] with $b = 2$.

Lemma 8. *Let* b *be a fixed integer greater than* 1*. Given a function* f *on* $[0,1]$*, suppose there exist two sequences of polynomials* $(g_n)_{\substack{n=b^k,\\ k\in\mathbb{Z}_+}}$*,* $(h_n)_{\substack{n=b^k,\\ k\in\mathbb{Z}_+}}$ *satisfying conditions (i), (iii), (iv) of Result 3 and the condition*

$$h_n(x) - g_n(x) = O(\Delta_n^{\alpha}(x)) \qquad \text{uniformly in } [0,1] \tag{6}$$

for $n = b^k$*,* $k \in \mathbb{Z}_+$*. Then these sequences can be augmented to full sequences* $(g_n)_{n\in\mathbb{Z}_+}$*,* $(h_n)_{n\in\mathbb{Z}_+}$ *satisfying conditions (i), (iii), (iv) from Result 3 and condition (6) for all* $n \in Z_+$*. In particular, then there exists an algorithm that simulates* f *at the rate* $O(\Delta_n^{\alpha}(x))$ *on* $[0,1]$*.*

Proof. Given the polynomials g_n and h_n for b-adic values of n, we will fill in the gaps in the two sequences in the obvious way: given n, let $n' := b^{\lfloor \log_b n \rfloor}$, and set

$$g_n(x,y) := (x+y)^{n-n'} g_{n'}(x,y), \qquad h_n(x,y) := (x+y)^{n-n'} h_{n'}(x,y).$$

The Bernstein coefficients of the obtained polynomials g_n, h_n are therefore some convex combinations of the coefficients of $g_{n'}$, $h_{n'}$. It follows that condition (i) holds for the full sequences (g_n), (h_n). It is clear from the construction that (iii) and (iv) hold as well, the latter condition being an equality except when jumping from one dyadic level to the next, when it is satisfied by our assumption. Recall that condition (ii) can always be satisfied, so there is no need to verify it explicitly. To check that (6) holds for the full sequences (g_n), (h_n), note that, by construction,

$$h_n(x) - g_n(x) = h_{n'}(x) - g_{n'}(x) = O(\Delta_{n'}^{\alpha}(x)).$$

But since $n' \le n < bn'$, we see that $O(\Delta_{n'}^{\alpha}(x)) = O(\Delta_n^{\alpha}(x))$. This completes the proof. ☐

We can now state our main result.

Theorem 9. *Let* $f : [0,1] \to (0,1)$*. If* $f \in C^{\alpha}[0,1]$*, then* f *can be simulated at the rate* $\Delta_n^{\alpha}(x)$ *on* $[0,1]$*. Precisely, there exist polynomials* g_n *and* f_n *satisfying conditions (i) – (iv) of Result 3 and bound (6). Conversely, if* f *is simulable at the rate* $\Delta_n^{\alpha}(x)$ *on the interval* $[0,1]$*, then* $f \in C^{\alpha}[0,1]$*.*

As noted already, Lorentz [3] stated a converse to Result 7 above, which (in a special case) can be written as follows.

Claim 10 ([3, Theorems 5 and 6]). *Let $\alpha > 0$. If a function f can be approximated by polynomials q_n of degree at most n with nonnegative Bernstein coefficients at the rate (5), then $f \in C^\alpha[0,1]$.*

The necessity argument in [3] skips technical details and refers to the work of Timan [8]. Specifically, we quote Theorem 6 from [3] and the subsequent discussion:

> " **Theorem 6.** *For each $r = 1, 2, \ldots$ there is a constant C_r with the following property. Let $\omega(h)$ be a modulus of continuity, and put*
>
> $$\tilde{\omega}(h) = h \int_h^1 \frac{\omega(u)}{u^2} du + \int_0^h \frac{\omega(u)}{u} du.$$
>
> *If $f(x)$ is a continuous function on $[0,1]$ and if there exists a sequence $P_n(x)$ of polynomials with positive coefficients of degree n such that*
>
> $$|f(x) - P_n(x)| \leq \Delta_n^r \omega(\Delta_n), \quad 0 \leq x \leq 1, \quad n = 0, 1, \ldots,$$
>
> *then f has on $[0,1]$ the continuous derivatives f', f'', …, $f^{(r)}$ and*
>
> $$\omega(f^{(r)}; h) \leq C_r \tilde{\omega}(h).$$
>
> We omit the proofs. The method of deriving theorems of this kind from inqualities of the Markov-Bernstein type is due essentially to S. Bernstein, and is well known. For the variation of it which fits the present situation especially well, compare [6, p. 357 and p.360]. It should be noted that [6, p. 357] contains an essential mistake: the derivative P'_{2m+1} on p.359 should have been estimated at a point different from x. However, the proof can be corrected. "

In this quote, [6] refers to the original Russian edition of Timan's work [8]. Trying to reconstruct Lorentz' complete argument for his Theorem 6, we came to the realization that this argument requires an extra assumption, in fact precisely the assumption of Bernstein-nonnegative consistent approximation, or equivalently, the nonnegative series representation (4) that is central to this paper. In the next section we show that, indeed, such a series representation of f with tails decaying at the rate Δ_n^α implies the C^α smoothness of the represented function f. Thus our results here also provide a correction to the statement of Lorentz. In Section 6, we construct a counterexample to the original necessity claim of Lorentz.

3 Consistent approximation implies smoothness in Theorem 9

Lorentz proved the following analogues of Bernstein's and Markov's inequalities (both original inequalities can be found, e.g., in [1]). This result of Lorentz is formulated for a certain class of functions Ω; we will use it only for the power functions $t \mapsto t^j$.

Result 11 ([3, Theorem 3]). *For each $r = 1, 2, \ldots$ and each $H > 0$ there is a constant $K_r = K_r(H)$ with the following property. If $\Omega(h)$ is an increasing positive function defined for all $h \geq 0$ such that*

$$\Omega(2h) \leq H\Omega(h), \quad h \geq 0,$$

then for each polynomial P_n with positive (Bernstein) coefficients of degree n, the inequality

$$P_n(x) \leq \Omega(\Delta_n(x)), \quad 0 \leq x \leq 1$$

implies

$$|P_n^{(r)}(x)| \leq K_r \Delta_n^{-r}(x)\,\Omega(\Delta_n(x)), \quad 0 \leq x \leq 1. \tag{7}$$

We need the following observation.

Lemma 12. *Let $0 < \beta \leq 1$. Suppose that the continuous function $\varphi : [0,1] \to \mathbb{R}$ satisfies*

$$|\varphi(y) - \varphi(x)| \leq \mathrm{const}\,|x-y|^\beta \tag{8}$$

for all $x, y \in [0,1]$ such that $0 < x < y \leq 2x$. Then (8) (with a different constant) holds for all $x, y \in [0,1]$, i.e., $\varphi \in C^\beta[0,1]$.

Proof. Given $0 < 2x < y \leq 1$, find k such that $2^k x < y \leq 2^{k+1} x$. We have

$$|\varphi(y) - \varphi(x)| \leq \sum_{i=1}^{k} \left|\varphi(2^i x) - \varphi(2^{i-1}x)\right| + \left|\varphi(y) - \varphi(2^k x)\right| \leq \mathrm{const} \sum_{i=1}^{k+1} (2^{i-1}x)^\beta\,.$$

Since $2^k x < y \leq 2(y-x)$, the right-hand side is bounded by a constant multiple of $|x-y|^\beta$. The case $x = 0$ in (8) then follows by continuity.

□

To prove the necessity of C^α-smoothness, we will follow the approach suggested by G. Lorentz in [3], which goes back to Timan [8] and ultimately to S. Bernstein.

Proof of necessity in Theorem 9. Suppose that f can be simulated at the rate Δ_n^α on the interval $[0,1]$. Using the sequence (g_n) that approximates f from below and satisfies the consistency requirement $(x+y)^n \tilde{g}_n(x,y) \preceq \tilde{g}_{2n}(x,y)$, we set $f_n := g_{2^{n+1}} - g_{2^n}$ and obtain the following nonnegative series representation for f:

$$f(x) = \sum_{n=0}^{\infty} f_n(x), \qquad f_n \in \mathrm{span}_+\{p_{2^n,k} : k = 0, \ldots, 2^n\}. \tag{9}$$

Here span_+ denotes the nonnegative span of a set. By the assumption on the rate of approximation, the polynomials f_n satisfy the bound

$$|f_n(x)| \leq \mathrm{const}\,\Delta_{2^n}^\alpha(x) \qquad \text{for all } x \in [0,1].$$

Now, the inequality (7) implies

$$|f_n^{(j)}(x)| \leq \mathrm{const}\,\Delta_{2^n}^{\alpha-j}(x) \qquad \text{for all } x \in [0,1],\ j \in \mathbb{N}. \tag{10}$$

This already ensures that $f^{(r)}$ is continuous in $[0,1]$. Our goal is to prove the inequality

$$\left|f^{(r)}(x) - f^{(r)}(y)\right| \leq \mathrm{const}\,|x-y|^{\alpha-r} \tag{11}$$

for $x, y \in [0,1]$, which means exactly that $f \in C^\alpha[0,1]$. By Lemma 12 applied to $\varphi = f^{(r)}$, it suffices to verify (11) under the assumption that $0 < x < y \le 2x$. We now write $f^{(r)}(x) - f^{(r)}(y)$ by splitting the sum (9) into two parts, below and above a certain threshold N (to be chosen later), and differentiating term by term:

$$f^{(r)}(x) - f^{(r)}(y) = \sum_{n=0}^{N-1} (f_n^{(r)}(x) - f_n^{(r)}(y)) + \sum_{n=N}^{\infty} (f_n^{(r)}(x) - f_n^{(r)}(y)).$$

Estimate the first sum using the bound (10) with $j = r+1$ and the second using the bound (10) with $j = r$. This yields

$$\begin{aligned}\left|f^{(r)}(x) - f^{(r)}(y)\right| &\le \sum_{n=0}^{N-1} |x-y||f_n^{(r)}(\xi_n)| + 2\sum_{n=N}^{\infty} \max\{|f_n^{(r)}(x)|, |f_n^{(r)}(y)|\} \\ &\le \operatorname{const}\Big(|x-y| \sum_{n=0}^{N-1} \Delta_{2^n}^{\alpha-r-1}(x) + 2\sum_{n=N}^{\infty} \Delta_{2^n}^{\alpha-r}(x)\Big) \qquad (12) \\ &\le \operatorname{const}|x-y|\Delta_{2^N}^{\alpha-r-1}(x) + \operatorname{const}\Delta_{2^N}^{\alpha-r}(x). \qquad (13)\end{aligned}$$

In obtaining (12) we used the inequalities $x < \xi_n < y \le 2x$ which imply that $\Delta_{2^n}(\xi_n) \le 2\Delta_{2^n}(x)$. Now choose N so that $\Delta_{2^{N+1}}(x) < |x-y| \le \Delta_{2^N}(x)$. Then the bound (13) yields (11). ◻

4 Lorentz operators and simultaneous approximation

We now recall the main ingredients of the valid proof of Result 7 (Theorem 1 from [3]). That proof is based on the Taylor expansion

$$f(x) = f\left(\frac{k}{n}\right) - \sum_{j=1}^{r} \frac{1}{j!}\left(\frac{k}{n} - x\right)^j f^{(j)}(x) + \frac{1}{r!}\left(\frac{k}{n} - x\right)^r [f^{(r)}(x) - f^{(r)}(\xi)], \qquad (14)$$

where ξ is a point between x and k/n and f is assumed to be r times differentiable. This formula is used in [3] to derive an asymptotic expansion of the ***Bernstein operator***

$$(B_n f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{nk}(x), \qquad (15)$$

where the polynomials p_{nk} are defined in (1). Lorentz uses expansion (14) to define operators Q_{nr} by the recurrence

$$\begin{aligned}(Q_{nr}f)(x) &:= (B_n f)(x) - \sum_{j=1}^{r} \frac{1}{j!n^j} T_{nj}(x)(Q_{n,r-j}f^{(j)})(x), \quad \text{where} \qquad (16) \\ T_{nj}(x) &:= \sum_{k=0}^{n} (k - nx)^j p_{nk}(x). \qquad (17)\end{aligned}$$

Note that the sum in (16) in fact starts at j=2 rather than at j=1, since the polynomial T_{n1} is identically zero. Also note that the expressions (16) must be written in the Bernstein basis of degree $n+r$, so that, e.g., the leading term $B_n f$ must be multiplied by the binomial expansion of $(x+(1-x))^r$ to appear in its Bernstein form of degree $n+r$. We will refer to the operators Q_{nr} mapping a function to a polynomial in Bernstein form of order $n+r$ as the ***Lorentz operators.*** As noted in [3], these operators can be rewritten as follows

$$(Q_{nr}f)(x) =: \sum_{k=0}^{n} \left(f\left(\frac{k}{n}\right) + \sum_{j=2}^{r} f^{(j)}\left(\frac{k}{n}\right) \frac{1}{n^j}\, \tau_{rj}(x,n) \right) p_{nk}(x) \tag{18}$$

The polynomials $\tau_{rj}(x,n)$ are shown in [3] to be of degree j in x and of degree $\lfloor j/2 \rfloor$ in n, and to be independent of f. We will need the following estimates on each polynomial τ_{rj}:

Lemma 13. *The polynomials* τ_{rj} *are bounded by*

$$|\tau_{rj}(x,n)| \le \mathrm{const}_j\ n^j \Delta_n(x)^j \qquad \text{for all } \ x \in [0,1], \tag{19}$$

where const_j *depends only on* j*.*

Proof. Formula (18) can be written more simply as

$$(Q_{nr}f)(x) = \sum_{k=0}^{n} \left(\sum_{j=0}^{r} f^{(j)}\left(\frac{k}{n}\right) \frac{1}{n^j} \tau_{rj}(x,n) \right) p_{nk}(x), \tag{20}$$

with the understanding that $\tau_{r0}(x,n) = 1$ and $\tau_{r1}(x,n) = 0$. Plugging (20) into (16), we obtain

$$(Q_{nr}f)(x) = \sum_{k=0}^{n} \left(f\left(\frac{k}{n}\right) - \sum_{j=2}^{r} f^{(j)}\left(\frac{k}{n}\right) \frac{1}{n^j} \sum_{l=2}^{j} \frac{1}{l!} T_{nl}(x,n) \tau_{r-l,j-l}(x,n) \right) p_{nk}(x).$$

By term-by-term comparison, this yields

$$\tau_{rj}(x,n) = -\sum_{l=2}^{j} \frac{1}{l!} T_{nl}(x) \tau_{r-l,j-l}(x,n) \qquad \text{for } \ j \ge 2. \tag{21}$$

By [3, p. 242],

$$|T_{nl}(x)| \le \mathrm{const}_l\ n^l \Delta_n^l(x)\,.$$

However, by induction,

$$|\tau_{r-l,j-l}(x,n)| \le \mathrm{const}_{j-l}\ n^{j-l} \Delta_n^{j-l}(x),$$

which gives (19), as required. □

Corollary 14. *Fix an integer* $r \ge 0$*. For any* $j \le r$*, write*

$$\tau_{rj}(x,n) = \sum_{i=0}^{j} a_i(n,j) x^i (1-x)^{j-i}\,. \tag{22}$$

Then for all $i \in [0,j]$*, we have* $a_i(n,j) = a_{j-i}(n,j)$ *and* $|a_i(n,j)| \le C_j^{\sharp} n^i$ *for some constants* $C_j^{\sharp}$*.*

Proof. The polynomials T_{nj} satisfy $T_{nj}(x) = T_{nj}(1-x)$, as is easily seen using the substitution $\tilde{k} = n-k$ in their definition (17). It then follows from the recursion (21) that $\tau_{rj}(x,n) = \tau_{rj}(1-x,n)$ as well, and this implies that $a_i(n,j) = a_{j-i}(n,j)$ for all i. Next, consider the polynomial $A(x) := \sum_{i=0}^{j} a_i(n,j)x^i$. Since $\tau_{rj}(x,n) = (1-x)^j A(\frac{x}{1-x})$, Lemma 13 implies that $|A(x)| \leq \mathrm{const}_j$ for $x \in [0, 1/n]$. Thus $A_*(x) := A(\frac{x+1}{2n})$ satisfies $|A_*(x)| \leq \mathrm{const}_j$ for $x \in [-1,1]$. Markov's inequality $\|A_*^{(i)}\|_\infty \leq j^{2i}\|A_*\|_\infty$ (see [1, Chapter 4, Theorem 1.4]) yields

$$|a_i(n,j)| = |A^{(i)}(0)| = (2n)^i|A_*^{(i)}(-1)| \leq C_j^\sharp n^i \text{ for all } i \leq j\,.$$

□

For our next argument, we will need an additional technical lemma that addresses the derivatives of the functions p_{nk}.

Lemma 15. *For any integer $\ell \geq 0$ and any $\beta \geq 0$,*

$$\sum_{k=0}^{n} \left|\frac{k}{n} - x\right|^\beta \left|p_{nk}^{(\ell)}(x)\right| \leq \mathrm{const}\, \Delta_n^{\beta-\ell}(x) \qquad \text{for all } x \in [0,1]. \tag{23}$$

Proof. The proof is by induction on ℓ. The proof for $\ell = 0$ is due to Lorentz [3, Lemma 1]. Our proof of the bound (23) for $\ell = 1$ splits into two cases.

<u>Case 1</u>. $\ell = 1$, $\Delta_n(x) = \sqrt{x(1-x)/n}$. In this case we substitute the equality

$$p'_{nk}(x) = \frac{k - nx}{x(1-x)} p_{nk}(x), \tag{24}$$

into the left-hand side of (23) to obtain

$$\begin{aligned}
\sum_{k=0}^{n} \left|\frac{k}{n} - x\right|^\beta \left|p'_{nk}(x)\right| &\leq \frac{\mathrm{const}}{x(1-x)} \sum_{k=0}^{n} \left|\frac{k}{n} - x\right|^\beta |k - nx|\, p_{nk}(x) = \frac{\mathrm{const}\; n}{x(1-x)} \sum_{k=0}^{n} \left|\frac{k}{n} - x\right|^{\beta+1} p_{nk}(x) \\
&\leq \mathrm{const}\, \frac{n}{x(1-x)} \Delta_n^{\beta+1}(x) = \mathrm{const}\, \Delta_n^{\beta-1}(x).
\end{aligned}$$

Here the last inequality uses the bound (23) already established for $\ell = 0$.

<u>Case 2</u>. $\ell = 1$, $\Delta_n(x) = 1/n$. In this case we substitute a different expression for $p'_{nk}(x)$, precisely

$$p'_{nk}(x) = np_{n-1,k-1}(x) - np_{n-1,k}(x), \tag{25}$$

which gives the bound

$$\begin{aligned}
\sum_{k=0}^{n} \left|\frac{k}{n} - x\right|^\beta \left|p'_{nk}(x)\right| &\leq \mathrm{const}\; n \sum_{k=0}^{n} \left|\frac{k}{n} - x\right|^\beta (p_{n-1,k-1}(x) + p_{n-1,k}(x)) \\
&\leq 2\,\mathrm{const}\; n \sum_{k=0}^{n-1} \left(\left|\frac{k}{n-1} - x\right|^\beta + \frac{1}{n}\left|\frac{k}{n-1} - x\right|^{\beta-1} \right) p_{n-1,k}(x) \\
&\leq \mathrm{const}\, n(\Delta_{n-1}^\beta(x) + \frac{1}{n-1}\Delta_{n-1}^{\beta-1}(x) \leq 2\,\mathrm{const}\,(n-1)^{-\beta+1}.
\end{aligned}$$

For large values of n, $n-1$ can be replaced by n, so this finishes the proof in Case 2.

For $\ell > 1$, the proof proceeds analogously. If $\Delta_n(x) = \sqrt{x(1-x)/n}$, we use the expression (24) to obtain

$$\begin{aligned}\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^{\beta}\left|p_{nk}^{(\ell)}(x)\right| &\leq \frac{\text{const}}{x(1-x)}\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^{\beta}|k-nx|\,|p_{nk}^{(\ell-1)}(x)| \\ &= \frac{\text{const}\ n}{x(1-x)}\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^{\beta+1}|p_{nk}^{(\ell-1)}(x)| \\ &\leq \text{const}\,\frac{n}{x(1-x)}\Delta_n^{\beta+2-\ell}(x) = \text{const}\,\Delta_n^{\beta-\ell}(x).\end{aligned}$$

If $\Delta_n(x) = 1/n$, we use the expression (25) to obtain

$$\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^{\beta}\left|(p'_{nk}(x))^{(\ell-1)}\right| \leq \sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^{\beta}\left(|p_{n-1,k-1}^{(\ell-1)}(x)|+|p_{n-1,k}^{(\ell-1)}(x)|\right)$$

and then finish the proof using the inductive assumption on $\ell-1$ as in Case 2.

This proves the bound (23) by induction. □

We now generalize Lemma 13 to derive bounds on the derivatives of the polynomials τ_{rj}.

Lemma 16. *The derivatives of the polynomials* T_{nj} *and* τ_{rj} *are bounded as follows*

$$\begin{aligned}|T_{nj}^{(\ell)}(x)| &\leq \text{const}_{j,\ell}\, n^j\Delta_n^{j-\ell}(x) && (26)\\ |\tau_{rj}^{(\ell)}(x,n)| &\leq \text{const}_{j,\ell}\, n^j\Delta_n^{j-\ell}(x) && (27)\end{aligned}$$

for all $x \in [0,1]$. *Here* $\text{const}_{j,\ell}$ *depends only on* j *and* ℓ.

Proof. Differentiate the formula (17) ℓ times to obtain

$$T_{nj}^{(\ell)}(x) = \sum_{m\leq\min j,\ell}\binom{\ell}{m}\sum_{k=0}^{n}(-n)^m(k-nx)^{j-m}p_{nk}^{\ell-m}(x).$$

By Lemma 15, each term is bounded by

$$n^m\cdot n^{j-m}\cdot\text{const}\,\Delta_n^{(j-m)-(\ell-m)}(x) = \text{const}\,n^j\Delta_n^{j-\ell}(x),$$

which proves the estimate (26). To get the analogous estimate for derivatives τ_{rj}, we run an inductive argument. Differentiating the formula (21) ℓ times, we get

$$\tau_{rj}^{(\ell)}(x,n) = -\sum_{m\leq\ell}\binom{\ell}{m}\sum_{s=2}^{j}\frac{1}{s!}T_{ns}^{(m)}(x)\tau_{r-s,j-s}^{\ell-m}(x,n). \qquad (28)$$

Applying the inductive assumption on the derivatives $\tau_{r-s,j-s}^{\ell-m}(x,n)$ and the already proven bound (26) on $T_{ns}^{(m)}(x)$, we obtain the estimate

$$\text{const}_{j,\ell}\,n^s\Delta_n^{s-m}(x)\,n^{j-s}\Delta_n^{j-s-\ell+m}(x) = \text{const}_{j,\ell}\,n^j\Delta_n^{j-\ell}(x)$$

on each term in the sum (28), and therefore on the function $|\tau_{rj}^{(\ell)}(x,n)|$ as well, proving (27). ∎

Next, we will show that the derivatives of polynomials $Q_{nr}f$ approximate the corresponding derivatives of f sufficiently well. This is known as ***simultaneous approximation***. Here is the precise result.

Theorem 17. *Let $f \in C^\alpha[0,1]$ and let $r := \lceil\alpha\rceil - 1$. Then, for any $j = 0, \ldots, r$,*

$$|((I - Q_{n,r})f)^{(j)}(x)| \leq \mathrm{const}_r \, \|f\|_{C^\alpha} \Delta_n^{\alpha-j}(x) \qquad \text{for all } \; x \in [0,1].$$

The case $j = 0$ of this theorem is contained in formula (22) of [3]. To prove Theorem 17 for $j \geq 1$, we first note that the Lorentz operator $Q_{n,r}$ reproduces polynomials of degree at most r, as the following lemma shows.

Lemma 18. *Let f be a polynomial of degree at most r. Then $Q_{n,r}f = f$.*

Proof. The proof is by induction on r. The result holds for $r = 0$ and 1 since $Q_{n,0}$ is simply the Bernstein operator, which reproduces linear functions. For higher values of r, the proof is as follows. The Taylor polynomial of f of degree r coincides with f, so

$$f(x) = f\left(\frac{k}{n}\right) - \sum_{j=1}^{r} \frac{1}{j!}\left(\frac{k}{n} - x\right)^j f^{(j)}(x),$$

so by multiplying by $p_{nk}(x)$ and summing over k, we obtain

$$f(x) = (B_n f)(x) - \sum_{j=1}^{r} \frac{1}{j! n^j} T_{nj}(x) f^{(j)}(x).$$

By our inductive assumption, $f^{(j)} = Q_{n,r-j}f^{(j)}$. Substituting this into (16), we get $Q_{n,r}f = f$. ∎

Proof of Theorem 17. Our goal is to show that the jth derivative of the difference between $Q_{n,r}f$ and f at any point x is bounded by a constant multiple of $\|f\|_{C^\alpha}\Delta_n^{\alpha-j}(x)$ regardless of x. Since $Q_{n,r}$ reproduces polynomials of degree r, we can subtract from f its Taylor polynomial of degree r centered at x without changing the difference $(Q_{n,r}f - f)^{(j)}(x)$. Thus, without loss of generality we can assume that the value of f and its derivatives up to order r are zero at x. Now, recall that

$$(Q_{n,r}f - f)^{(j)}(x) = \left(\sum_{k=0}^{n}\left(f\left(\frac{k}{n}\right) - f(x)\right)p_{nk}(x) + \sum_{i=2}^{r}\sum_{k=0}^{n} f^{(i)}\left(\frac{k}{n}\right)\frac{1}{n^i}\tau_{ri}(x,n)p_{nk}(x)\right)^{(j)}.$$

Differentiating these sums j times, we will obtain terms of two kinds. Terms of the first kind are obtained from differentiating the first sum; they have the form

$$\sum_{k=0}^{n}\left(f\left(\frac{k}{n}\right) - f(x)\right)^{(\ell)} p_{nk}^{(j-\ell)}(x)$$

for some ℓ between 0 and j. Each of these sums can be bounded as follows, using Lemma 15:

$$\begin{aligned}\left|\sum_{k=0}^{n}\left(f\left(\frac{k}{n}\right)-f(x)\right)^{(\ell)}p_{nk}^{(j-\ell)}(x)\right| &\le \sum_{k=0}^{n}\|f\|_{C^\alpha}\left|\frac{k}{n}-x\right|^{\alpha-\ell}|p_{nk}^{(j-\ell)}(x)| \\ &\le \text{const}\,\|f\|_{C^\alpha}\Delta_n^{(\alpha-\ell)-(j-\ell)}(x) \\ &= \text{const}\,\|f\|_{C^\alpha}\Delta_n^{\alpha-j}(x).\end{aligned}$$

Terms of the second kind are obtained by differentiating any of the other sums for $i = 2,\dots,r$ and have the form

$$\sum_{k=0}^{n} f^{(i)}\left(\frac{k}{n}\right)\frac{1}{n^i}\tau_{ri}^{(\ell)}(x,n)p_{nk}^{(j-\ell)}(x)$$

for some ℓ between 0 and j. Taking into account that the derivatives of f up to order r vanish at x, each of these sums can be bounded by

$$\left|\sum_{k=0}^{n} f^{(i)}\left(\frac{k}{n}\right)\frac{1}{n^i}\tau_{ri}^{(\ell)}(x,n)p_{nk}^{(j-\ell)}(x)\right| \le \sum_{k=0}^{n}\|f\|_{C^\alpha}\left|\frac{k}{n}-x\right|^{\alpha-i}\frac{1}{n^i}|\tau_{ri}^{(\ell)}(x,n)|\left|p_{nk}^{(j-\ell)}(x)\right|.$$

Invoking the bound (27) from Lemma 16 on the terms $|\tau_{ri}^{(l)}(x,n)|$, we conclude that the total is bounded by

$$\text{const}\,\|f\|_{C^\alpha}\frac{1}{n^i}\,n^i\Delta^{i-\ell}(x)\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^{\alpha-i}|p_{nk}^{(j-\ell)}(x)|.$$

The last sum, in turn, is estimated according to Lemma 15 to produce the final bound

$$\text{const}\,\|f\|_{C^\alpha}\Delta_n^{i-\ell}(x)\cdot\Delta_n^{\alpha-i-j+\ell}(x) = \text{const}\,\|f\|_{C^\alpha}\Delta_n^{\alpha-j}(x).$$

This completes the proof. □

Theorem 19. *Let $f \in C^\alpha[0,1]$ and let $r := \lceil\alpha\rceil - 1$. Then the following bounds hold for $x \in [0,1]$:*

$$\begin{aligned}|\left(Q_{n,r}f-f\right)^{(r)}(x)| &\le \text{const}\,\Delta_n^{\alpha-r}(x)\|f\|_{C^\alpha} &(29)\\ |\left(Q_{n,r}f\right)^{(r+1)}(x)| &\le \text{const}\,\Delta_n^{\alpha-r-1}(x)\|f\|_{C^\alpha} &(30)\\ \|Q_{n,r}f\|_{C^\alpha} &\le \text{const}\,\|f\|_{C^\alpha} &(31)\end{aligned}$$

with constants independent of f and n.

Proof. The bound (29) is a special case of Theorem 17 with $j = r$. To establish the other two bounds, we may assume without loss of generality that f vanishes to order r at x, since polynomials of degree at most r are reproduced by $Q_{n,r}$ and then annihilated by taking the derivative of order $r+1$, as well as by taking the rth derivative followed by a difference at two points x and y.

The assumption made above implies that for all $i \le r$,

$$|f^{(i)}(k/n)| \le \text{const}\,\|f\|_{C^\alpha}|k/n-x|^{\alpha-i}. \qquad (32)$$

By direct differentiation,

$$(Q_{n,r}f)^{(r+1)}(x) = \sum_{i=0}^{r}\sum_{\ell=0}^{r+1}\binom{r+1}{\ell}\sum_{k=0}^{n}f^{(i)}\left(\frac{k}{n}\right)\frac{1}{n^i}\tau_{ri}^{\ell}(x,n)p_{nk}^{r+1-\ell}(x). \qquad (33)$$

Fix $i \in [0,r]$ and $\ell \in [0,r+1]$. The summand corresponding to i and ℓ in (33) can be bounded by

$$\left|\sum_{k=0}^{n} f^{(i)}\left(\frac{k}{n}\right)\frac{1}{n^i}\tau_{ri}^{(\ell)}(x,n)p_{nk}^{(r+1-\ell)}(x)\right| \leq \sum_{k=0}^{n}\|f\|_{C^\alpha}\left|\frac{k}{n}-x\right|^{\alpha-i}\frac{1}{n^i}|\tau_{ri}^{(\ell)}(x,n)|\,\left|p_{nk}^{(r+1-\ell)}(x)\right|. \tag{34}$$

Invoking Lemma 13, we note that the terms $\frac{1}{n^i}|\tau_{ri}^{(\ell)}(x,n)|$ are bounded by a constant multiple of $\Delta_n^{i-\ell}(x)$, therefore (34) is bounded by

$$\text{const}\,\|f\|_{C^\alpha}\Delta_n^{i-\ell}(x)\Delta_n^{\alpha-i-r-1+\ell}(x) = \text{const}\,\|f\|_{C^\alpha}\Delta_n^{\alpha-r-1}(x).$$

This proves (30).

To establish the remaining bound (31), we need to estimate the expression

$$\left|(Q_{n,r}f)^{(r)}(x) - (Q_{n,r}f)^{(r)}(y)\right| \tag{35}$$

for two points x and y in $[0,1]$. By Lemma 12, we may assume that $0 < x < y \leq 2x$. Consider two cases.

<u>Case 1</u>. If $|x-y| \geq \Delta_n(x)$, then we estimate (35) using the triangle inequality and the bound (29) on each of the two terms, $(Q_{n,r}f - f)^{(r)}(x)$ and $(Q_{n,r}f - f)^{(r)}(y)$. Altogether, this bounds (35) from above by

$$\text{const}\,\|f\|_{C^\alpha}\Delta_n^{r-\alpha}(x) \leq \text{const}\,\|f\|_{C^\alpha}|x-y|^{r-\alpha}.$$

<u>Case 2</u>. If $|x-y| \leq \Delta_n(x)$, then $|x-y| \leq \Delta_n^{r+1-\alpha}|x-y|^{\alpha-r}$ so for some $\xi \in (x,y)$,

$$\begin{aligned}\left|(Q_{n,r}f)^{(r)}(x) - (Q_{n,r}f)^{(r)}(y)\right| &= (Q_{n,r}f)^{(r+1)}(\xi)\cdot|x-y| \\ &\leq \text{const}\,\Delta_n^{\alpha-r-1}(x)\|f\|_{C^\alpha}\cdot\Delta_n^{r+1-\alpha}(x)|x-y|^{\alpha-r}.\end{aligned}$$

This establishes (31), and completes the proof of the theorem. □

Lemma 20. *Suppose that $f : [0,1] \to \mathbb{R}$ satisfies $|f^{(r+1)}(x)| \leq \Delta_n^{-\beta}(x)$ for some $\beta \in [0,1]$ and all $x \in [0,1]$. Then for all $x \in [0,1]$, we have*

$$\left|\Big(Q_{n,r}f\Big)^{(r+1)}(x)\right| \leq \text{const}\,\Delta_n^{-\beta}(x). \tag{36}$$

with a constant independent of f and n.

Proof. We start by proving a simple inequality valid for all $x, \xi \in [0,1]$:

$$\frac{\Delta_n(x)}{\Delta_n(\xi)} \leq 2\Big(1 + \frac{|x-\xi|}{\Delta_n(x)}\Big). \tag{37}$$

By the symmetry $\Delta_n(x) = \Delta_n(1-x)$, we may assume that $x, \xi \in [0,1/2]$. We also assume that $\xi < x$ and $\Delta_n(x) > 1/n$, since otherwise the inequality is obvious. If $\xi \geq x/2$ then the left-hand side of (37) is at most 2, so we may assume that $\xi < x/2$. In this case we have

$$\Delta_n^2(x) \leq x/n \leq 2|x-\xi|/n \leq 2|x-\xi|\,\Delta_n(\xi),$$

which implies (37). To prove (36), we may assume as in the preceding theorem that f vanishes to order r at x. This implies that for all $i \le r$ and $z \ne x$ in $[0,1]$, there exists ξ between x and z such that

$$\frac{|f^{(i)}(z)|}{|z-x|^{r+1-i}} \le |f^{(r+1)}(\xi)| \le \Delta_n^{-\beta}(\xi) \le 2\Delta_n^{-\beta}(x)\Big(1+\frac{|x-z|}{\Delta_n(x)}\Big)\,, \tag{38}$$

where the last step used (37) taken to the power β, and the inequality $|x-\xi| \le |x-z|$.

Recall the expression (33) for $(Q_{n,r}f)^{(r+1)}(x)$. Fix $i \in [0,r]$ and $\ell \in [0,r+1]$. The summand

$$\left|\sum_{k=0}^{n} f^{(i)}\left(\frac{k}{n}\right)\frac{1}{n^i}\tau_{ri}^{(\ell)}(x,n)p_{nk}^{(r+1-\ell)}(x)\right|$$

corresponding to i and ℓ in (33), can be bounded using (38) and Lemma 13 by

$$\mathrm{const}\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^{r+1-i}\Delta_n^{-\beta}(x)\Big(1+\frac{|\frac{k}{n}-x|}{\Delta_n(x)}\Big)\,\Delta_n^{i-\ell}(x)\,\left|p_{nk}^{(r+1-\ell)}(x)\right|. \tag{39}$$

Invoking Lemma 15 twice, we conclude that (39) is bounded by

$$\mathrm{const}\,\Delta_n^{-\beta}(x)\Delta_n^{i-\ell}(x)\Big(\Delta_n^{\ell-i}(x)+\frac{\Delta_n^{\ell-i+1}(x)}{\Delta_n(x)}\Big) \le \mathrm{const}\,\Delta_n^{-\beta}(x)\,.$$

This proves the lemma. ∎

5 The iterative construction

Now we embark on the proof of the sufficiency in Theorem 9, that is, we show that any function $f \in C^\alpha[0,1]$ can be simulated at the rate Δ_n^α on $[0,1]$. Below is a sketch of this multi-step proof. We may and shall assume that

$$\|f\|_{C^\alpha} = 1\,. \tag{40}$$

We construct, for a fixed integer $b>1$, two sequences (g_n) and (h_n), indexed by $n=b^\ell$, $\ell \in \mathbb{Z}_+$ and satisfying conditions (i), (ii), (iv), (6). (For technical reasons, the degree of g_n and h_n will be $n+r$ instead of n; since r is constant, this will be immaterial.) Recall that such a construction proves the Theorem due to Reduction Lemma 8.

Our construction is inductive. We assume that we have approximants $g_{n/b}$ and $h_{n/b}$ of degree at most $n/b+r$ that satisfy conditions (i), (ii), (iv), (6) and the additional bounds

$$\begin{aligned} c\Delta_{n/b}^\alpha(x) \le h_{n/b}(x)-f(x) &\le C\Delta_{n/b}^\alpha(x),\\ c\Delta_{n/b}^\alpha(x) \le f(x)-g_{n/b}(x) &\le C\Delta_{n/b}^\alpha(x) \end{aligned} \tag{41}$$

for all $x \in [0,1]$, with appropriately chosen constants c and C; we then show that new approximants g_n, h_n of degree at most $n+r$ can be found so that they satisfy the same conditions as well as

$$\begin{aligned} c\Delta_n^\alpha(x) \le h_n(x)-f(x) &\le C\Delta_n^\alpha(x),\\ c\Delta_n^\alpha(x) \le f(x)-g_n(x) &\le C\Delta_n^\alpha(x) \end{aligned} \tag{42}$$

with the constants c, C unchanged.

We will focus on the construction of the sequence (g_n), since the construction of (h_n) is completely parallel, with appropriate changes of sign. Throughout, n will run along powers of b. In addition to (42), we will require that

$$g_n(x) - g_{n/b}(x) \quad \text{is a Bernstein polynomial of degree } n+r \text{ with nonnegative coefficients} \tag{43}$$

(this is equivalent to the consistent approximation inequality, $(x+y)^{n-n/b}\tilde{g}_{n/b}(x,y) \preceq \tilde{g}_n(x,y)$).

Define

$$\Gamma_n(x) := \Big(\frac{x+1/n}{n}\Big)^{1/2} \qquad \text{and} \qquad \Upsilon_n(x) := \Gamma_n(x-x^2)\,, \tag{44}$$

and observe that $\Delta_n(x) \le \Upsilon_n(x) \le 2\Delta_n(x)$ for all $x \in [0,1]$. Let $g_1 := 0$.

Our main step is the following recursive choice of g_n for $n \ge b$:

$$g_n := g_{n/b} + Q_{n,r}(f - g_{n/b} - D\Upsilon^{\alpha}_{n/b}). \tag{45}$$

This recursion defines g_n as a Bernstein polynomial of degree $n+r$ for every $n \ge b$ that is a power of b. The choice of the integer b as well as the constants c, C, and D will be made later to satisfy (42) and (43). This will (eventually) complete the inductive step.

Our first goal is to show that $\|g_n\|_{C^\alpha} \le \text{const}$, with const independent of n. The next lemma controls the contribution of the correction term Υ^α_n.

Lemma 21. *For any $j \ge 1$ and $x \in [0,1]$, we have $\Big|\Big(\Gamma^\alpha_n\Big)^{(j)}(x)\Big| \le C_j \Gamma_n^{\alpha-j}(x)$ and consequently $\Big|\Big(\Upsilon^\alpha_n\Big)^{(j)}(x)\Big| \le C^*_j \Upsilon_n^{\alpha-j}(x) \le \widetilde{C}_j \Delta_n^{\alpha-j}(x)$. Moreover,*

$$\Gamma_*(\alpha) := \sup_n \|\Gamma^\alpha_n\|_{C^\alpha} < \infty \quad \text{and} \quad \Upsilon_*(\alpha) := \sup_n \|\Upsilon^\alpha_n\|_{C^\alpha} < \infty\,, \tag{46}$$

where the norms are taken in $[0,1]$.

Proof. We start with the claims involving Γ^α_n. The upper bound on the jth derivative follows from the inequality

$$(1/n)^{\alpha/2}(x+1/n)^{\alpha/2-j} \le (1/n)^{(\alpha-j)/2}(x+1/n)^{(\alpha-j)/2},$$

which can be verified by direct inspection. To bound the C^α norm, let $r := \lfloor \alpha \rfloor$ and calculate

$$\Big|\Big(\Gamma^\alpha_n\Big)^{(r)}(x+h) - \Big(\Gamma^\alpha_n\Big)^{(r)}(x)\Big| = c_r n^{-\alpha/2}|(x+h+1/n)^{\alpha/2-r} - (x+1/n)^{\alpha/2-r}|\,.$$

For fixed h the right-hand side is maximized by taking $x=0$. By considering separately the cases $h > 1/n$ and $h \le 1/n$, one verifies that

$$n^{-\alpha/2}|(h+1/n)^{\alpha/2-r} - (1/n)^{\alpha/2-r}| \le ch^{\alpha-r}\,. \tag{47}$$

Indeed, the case $h > 1/n$ is immediate, while in the case $h \le 1/n$ we can bound the left-hand side of (47) using the mean value theorem by $cn^{-\alpha/2}h(1/n)^{\alpha/2-r-1}$ which is indeed at most $ch^{\alpha-r}$.

The chain rule and induction imply that the jth derivative of Υ_n^α can be represented in the form

$$\left(\Upsilon_n^\alpha\right)^{(j)}(x)=\sum_{i=0}^{j}\psi_i(x)\left(\Gamma_n^\alpha\right)^{(i)}(x-x^2)$$

where ψ_i are polynomials that do not depend on n. This yields the two assertions involving Υ_n^α.

$\square$

Theorem 22. *Suppose $\alpha\in(r,r+1)$, and let the sequence (g_n) be defined by (45), with an arbitrary constant D. Then, for b sufficiently large and n which is a power of b, we have*

$$\sup_n \|f-g_n\|_{C^\alpha}\le M \tag{48}$$

where the norm is taken in $[0,1]$ and $M=M(\alpha)<\infty$.

Abbreviate $Q_n:=Q_{n,r}$. The proof will use two estimates that we have already established for all $\alpha\in[r,r+1]$: the estimate (29), which we now rewrite as

$$|((I-Q_n)\phi)^{(r)}(x)|\le c_1\Delta_n^{\alpha-r}(x)\|\phi\|_{C^\alpha}, \tag{49}$$

and the hypercontractive estimate (30), which we rewrite as

$$|(Q_n\phi)^{(r+1)}(x)|\le c_2\Delta_n^{-\delta}(x)\|\phi\|_{C^\alpha},\quad \text{where }\ \delta:=r+1-\alpha. \tag{50}$$

Here the constants c_1,c_2 depend only on r. Enlarging c_2 if necessary, we assume that

$$c_2\ge 2\widetilde{C}_{r+1} \tag{51}$$

where $\widetilde{C}_{r+1}$ is the constant appearing in Lemma 21.

Recall that by combining (49) and (50) we derived in Theorem 19 the bound

$$\|(I-Q_n)\phi\|_{C^\alpha}\le c_3\|\phi\|_{C^\alpha}\,. \tag{52}$$

Lemma 23. *Let $\alpha\in(r,r+1)$. If b is sufficiently large and n is a power of b, then*

$$|g_n^{(r+1)}(x)|\le 2c_2\Delta_n^{-\delta}(x)\,,$$

Remark. The precise assumption on b that we will use is

$$2b^{-\delta/2}(c_4+1)<1/2\,, \tag{53}$$

where c_4 denotes the constant from Lemma 20. Note that this requires that $\delta>0$, i.e., $\alpha<r+1$.

Proof of Lemma 23. We argue by induction. The inequality certainly holds for $g_b=0$. We state the induction hypothesis

$$|g_{n/b}^{(r+1)}(x)|\le 2c_2\Delta_{n/b}^{-\delta}(x)\|f\|_{C^\alpha}\,. \tag{54}$$

The recursion $g_n=g_{n/b}+Q_n(f-g_{n/b}-D\Upsilon_{n/b}^\alpha)$ from (45) yields

$$|g_n^{(r+1)}(x)|\le|((I-Q_n)g_{n/b})^{(r+1)}(x)|+|(Q_nf)^{(r+1)}(x)|+\left|\left(\Upsilon_{n/b}^\alpha\right)^{(r+1)}(x)\right|. \tag{55}$$

The second term in (55) is bounded using (50). The third term is at most $\widetilde{C}_{r+1}\Delta_n^{-\delta}(x) \leq \frac{c_2}{2}\Delta_n^{-\delta}(x)$ by Lemma 21 and (51). To estimate the first term, note that

$$\Delta_{n/b}(x) \geq \sqrt{b}\Delta_n(x),$$

hence our inductive assumption (54) implies

$$|g_{n/b}^{(r+1)}(x)| \leq 2c_2 b^{-\delta/2}\Delta_n^{-\delta}(x)\,.$$

Apply Lemma 20; recall that the constant there is now denoted by c_4. We conclude that the first term in (55) is bounded by

$$|((I - Q_n)g_{n/b})^{(r+1)}(x)| \leq 2c_2(1 + c_4)b^{-\delta/2}\Delta_n^{-\delta}(x), \tag{56}$$

so the total is bounded as follows

$$|g_n^{(r+1)}(x)| \leq \Big(2b^{-\delta/2}(c_4 + 1) + 1 + 1/2\Big)\, c_2\Delta_n^{-\delta}(x)\,.$$

The right-hand side is at most $2c_2\Delta_n^{-\delta}(x)$ by the assumption (53). This proves the lemma.

□

Proof of Theorem 22. We can finally prove (48) by induction for n which is a power of b. Write

$$\Psi_n := (I - Q_n)(f - g_{n/b}) \quad \text{so that} \quad f - g_n = \Psi_n + DQ_n\Upsilon_{n/b}^{\alpha}\,. \tag{57}$$

By (31) and Lemma 21, we have

$$\|Q_n\Upsilon_{n/b}^{\alpha}\|_{C^\alpha} \leq \text{const}\,\|\Upsilon_{n/b}^{\alpha}\|_{C^\alpha} \leq c^* \tag{58}$$

for some $c^* = c^*(\alpha)$, so we focus on bounding the C^α norm of Ψ_n. The induction hypothesis $\|f - g_{n/b}\|_{C^\alpha} \leq M$, in conjunction with (49), implies that

$$|\Psi_n^{(r)}(x)| = |((I - Q_n)(f - g_{n/b}))^{(r)}(x)| \leq c_1 M\Delta_n^{\alpha - r}(x)\,.$$

Recall that it is enough to argue the C^α-smoothness of Ψ_n in the case $x < y \leq 2x$ due to Lemma 12. For $|x - y| \geq A\Delta_n(x)$, the triangle inequality implies

$$\frac{|\Psi_n^{(r)}(x) - \Psi_n^{(r)}(y)|}{|x - y|^{\alpha - r}} \leq \frac{2c_1 M}{A^{\alpha - r}}\,. \tag{59}$$

On the other hand, for $|x - y| \leq A\Delta_n(x)$ we use (56) to write

$$\begin{aligned}\frac{|((I - Q_n)g_{n/b})^{(r)}(x) - ((I - Q_n)g_{n/b})^{(r)}(y)|}{|x - y|^{\alpha - r}} &\leq 2c_2(c_4 + 1)b^{-\delta/2}\Delta_n^{-\delta}(x)|x - y|^\delta \\ &\leq 2c_2(c_4 + 1)b^{-\delta/2}A^\delta\,.\end{aligned}$$

Combining this with the inequality $\|(I - Q_n)f\|_{C^\alpha} \leq c_3$ (a consequence of (52) and (40)) and the definition of Ψ_n, we conclude that for $|x - y| \leq A\Delta_n(x)$,

$$\frac{|\Psi_n^{(r)}(x) - \Psi_n^{(r)}(y)|}{|x - y|^{\alpha - r}} \leq \Big(c_3 + 2c_2(c_4 + 1)b^{-\delta/2}A^\delta\Big)\,.$$

If $A^{\alpha - r} > 4c_1$ (this needs the assumption $\alpha > r$) and $M \geq c_3 + 2c_2(c_4 + 1)b^{-\delta/2}A^\delta + 2Dc^*$, then (59), (58) and the last display yield (48). □

Corollary 24. *Under the assumptions of Theorem 22, there exist constants $E_r, M_r < \infty$ such that for all $j = 0, \ldots, r$ and $x \in [0,1]$,*

$$\sup_n |(f - g_n)^{(j)}(x)| \le (E_r + DM_r)\,\Delta_n^{\alpha-j}(x). \tag{60}$$

Proof. For any $j = 0, \ldots, r$, note that

$$(f - g_n)^{(j)} = \Big((I - Q_n)(f - g_{n/b} - D\Upsilon^\alpha_{n/b}) + D\Upsilon^\alpha_{n/b}\Big)^{(j)}.$$

By Theorem 17,

$$\begin{aligned}\Big((I - Q_n)(f - g_{n/b} - D\Upsilon^\alpha_{n/b})\Big)^{(j)} &\le \mathrm{const}_r\,\|f - g_{n/b} - D\Upsilon^\alpha_{n/b}\|_{C^\alpha}\Delta_n^{\alpha-j}(x)\\ &\le \mathrm{const}_r(M + D\Upsilon_*(\alpha))\Delta_n^{\alpha-j}(x)\,, \end{aligned} \tag{61}$$

where the last step uses Theorem 22 and (46). Moreover, by Lemma 21,

$$\Big|\Big(\Upsilon^\alpha_{n/b}\Big)^{(j)}(x)\Big| \le \widetilde{C}_j\Delta_{n/b}^{\alpha-j}(x) \le \widetilde{C}_j b^{\alpha-j}\Delta_n^{\alpha-j}(x)\,.$$

Denoting $E_r = \mathrm{const}_r\, M$ and $M_r = \mathrm{const}_r\, \Upsilon_*(\alpha) + \max_{j\le r}\widetilde{C}_j b^{\alpha-j}$, we obtain the bound (60). □

Proof of Sufficiency in Theorem 9. From the sketch of the proof that precedes Theorem 22, recall that we need to verify that the integer b and the constants c, C, and D can be chosen so that the new term g_n from (45) meets requirements (42) and (43). We now reduce these requirements to a simple set of inequalities on the constants involved that can be all satisfied simultaneously.

To analyze the requirement (43) on the difference $g_n(x) - g_{n/b}(x)$, we use (20) to write it as

$$\begin{aligned} g_n(x) - g_{n/b}(x) &= Q_n(f - g_{n/b} - D\Upsilon^\alpha_{n/b})(x)\\ &= \sum_{j=0}^{r} n^{-j}\sum_{k=0}^{n}\Big(f - g_{n/b} - D\Upsilon^\alpha_{n/b}\Big)^{(j)}\left(\frac{k}{n}\right)\tau_{rj}(x,n)p_{n,k}(x)\\ &= \sum_{j=1}^{r}\Phi_j(x)\,, \end{aligned}$$

where $\Phi_j(x)$ is a polynomial of degree $n + j$ in Bernstein form

$$\Phi_j(x) := \frac{1}{r}(x + (1-x))^j\sum_{k=0}^{n}\Big(f - g_{n/b} - D\Upsilon^\alpha_{n/b}\Big)\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} + n^{-j}\Lambda_j(x)$$

and

$$\Lambda_j(x) := \sum_{k=0}^{n}\Big(f - g_{n/b} - D\Upsilon^\alpha_{n/b}\Big)^{(j)}\left(\frac{k}{n}\right)\tau_{rj}(x,n)\binom{n}{k}x^k(1-x)^{n-k}\,.$$

We will show that Φ_j has positive coefficients by bounding from above the coefficients of Λ_j. By symmetry, it suffices to consider the coefficient of $x^k(1-x)^{n+j-k}$ for $k \le (n+j)/2$. Two auxiliary inequalities we will use for this purpose, which are easily verified for $k < 2n/3$, are

$$\binom{n}{k-i} \le \left(\frac{3k}{n}\right)^i\binom{n}{k} \tag{62}$$

and

$$k^{j/2}\Delta_{n/b}^{-j}\Big(\frac{k}{n}\Big) \le 6^j k^{j/2}\Gamma_{n/b}^{-j}\Big(\frac{k}{n}\Big) \le (6n)^j b^{-j/2}\,. \tag{63}$$

Also, recall from Corollary 14 that $\tau_{rj}(x,n) = \sum_{i=0}^{j} a_i(n,j)x^i(1-x)^{j-i}$, where for all $i \in [0,j]$, we have $|a_i(n,j)| \le C_j^\sharp n^{\min\{i,j-i\}}$.

For $k \le (n+j)/2$, the coefficient of $x^k(1-x)^{n+j-k}$ in Λ_j can be bounded as follows:

$$\begin{aligned}
&\Big|\sum_{i=0}^{j}\Big(f - g_{n/b} - D\Upsilon_{n/b}^{\alpha}\Big)^{(j)}\left(\frac{k-i}{n}\right)a_i(n,j)\binom{n}{k-i}\Big| \\
\le\ & \sum_{i=0}^{j}(E_r + DM_r + D\widetilde{C}_j)\Delta_{n/b}^{\alpha-j}\left(\frac{k-i}{n}\right)C_j^\sharp n^{i\wedge(j-i)}\Big(\frac{3k}{n}\Big)^{i}\binom{n}{k} \\
\le\ & (j+1)(E_r + DM_r + D\widetilde{C}_j)2^r\Delta_{n/b}^{\alpha-j}\left(\frac{k}{n}\right)C_j^\sharp n^{j/2}\Big(\frac{3k}{n}\Big)^{j/2}\binom{n}{k} \\
\le\ & (\widetilde{E}_r + D\widetilde{M}_r)b^{-j/2}\Delta_{n/b}^{\alpha}\left(\frac{k}{n}\right)n^j\binom{n}{k},
\end{aligned} \tag{64}$$

where $\widetilde{E}_r := 2^r \max_{1\le j\le r} 18^j E_r(j+1)C_j^\sharp$ and $\widetilde{M}_r := 2^r \max_{1\le j\le r} 18^j (M_r + \widetilde{C}_j)(j+1)C_j^\sharp$. In the first step of the preceding display, we applied Corollary 24, Lemma 21 and (62); in the second step, we bounded a sum of $j+1$ terms by $j+1$ times the largest term, and in the last step we applied the inequality (63).

Therefore, by the definition of Φ_j and (41), the coefficient of $x^k(1-x)^{n+j-k}$ in Φ_j is at least

$$\Big(\frac{c}{r} - 2^\alpha D\Big)\Delta_{n/b}^{\alpha}\left(\frac{k}{n}\right)\binom{n}{k} - (\widetilde{E}_r + D\widetilde{M}_r)b^{-j/2}\Delta_{n/b}^{\alpha}\left(\frac{k}{n}\right)\binom{n}{k}$$

Thus nonnegativity of the Bernstein coefficients of all Φ_j is assured if

$$\frac{c}{r} - 2^\alpha D - (\widetilde{E}_r + D\widetilde{M}_r)b^{-1/2} \ge 0. \tag{65}$$

Our next task is to make sure that the upper bound in (42) holds. This is implied by Corollary 24 with $j = 0$ provided that

$$E_r + DM_r \le C\,. \tag{66}$$

The final condition is the lower bound in (42). Using the expression

$$f - g_n = D\Upsilon_{n/b}^{\alpha} + (I - Q_n)(f - g_{n/b} - D\Upsilon_{n/b}^{\alpha})\,,$$

the triangle inequality, and the pointwise upper bound

$$|(I - Q_n)(f - g_{n/b} - D\Upsilon_{n/b}^{\alpha})(x)| \le \mathrm{const}_r(M + D\Upsilon_*(\alpha))\Delta_n^\alpha(x)\,,$$

from (61), we obtain, for all $x \in [0,1]$, the inequality

$$\begin{aligned}
f(x) - g_n(x) \ &\ge\ D\Delta_{n/b}^{\alpha} - \mathrm{const}_r(M + D\Upsilon_*(\alpha))\Delta_n^\alpha(x) \\
&\ge\ \big(Db^{1/2} - \mathrm{const}_r(M + D\Upsilon_*(\alpha))\big)\Delta_n^\alpha(x)\,.
\end{aligned} \tag{67}$$

By requiring that the right-hand side of (67) be greater than $c\Delta_n^\alpha(x)$, we will guarantee that the lower bound in (42) holds. Thus we must satisfy

$$Db^{1/2} \geq \mathrm{const}_r(M + D\Upsilon_*(\alpha)) + c\,. \tag{68}$$

Finally, it is easily seen that all three conditions (65), (66) and (68) can be met simultaneously by first taking c so that the lower bound in (42) holds for $n = 1$, then picking $D = c/(r2^{\alpha+1})$, setting C to satisfy (66) as well as the upper bound in (42) for $n = 1$, and finally picking b large enough to satisfy (65) and (68). This justifies the inductive step and concludes the proof of the theorem. ∎

6 Revisiting the claim of Lorentz

The goal of this section is to demonstrate that Lorentz' Claim 10 made in [3] is invalid. Our counterexample will be constructed in several steps. We begin with some elementary observations about Bernstein polynomials.

Lemma 25. *Let* $B_n[a,b] := \{\sum_{k=0}^n c_k(x-a)^k(b-x)^{n-k} : c_k \geq 0\}$*. Then*

(a) $B_n[a,b] \subset B_{n+1}[a,b]$,

(b) $B_n[a,b] \cdot B_m[a,b] \subset B_{n+m}[a,b]$,

(c) $B_n[a,b] \subset B_n[c,d]$ *for every subinterval* $[c,d]$ *of the interval* $[a,b]$,

(d) B_n *is a convex cone of functions.*

Proof.
(a) Multiply by $1 = \frac{1}{b-a}[(x-a)+(b-x)]$ and distribute.
(b) Multiply out.
(c)

$$x - a = (c-a) + (x-c) \in B_0[c,d] + B_1[c,d] = B_1[c,d]\ ,$$
$$b - x = (b-d) + (d-x) \in B_0[c,d] + B_1[c,d] = B_1[c,d]\ .$$

Hence,

$$(x-a)^k(b-x)^{n-k} \in B_1[c,d]^n \subset B_n[c,d]\ .$$

(d) Obvious. ∎

Lemma 26. *Suppose that* p *is a polynomial of degree* n *with real coefficients such that* $p(0) > 0$ *and* p *has no roots in the unit disc* $\{|z| \leq 1\}$*. Then* $p \in B_n[-1,1]$.

Proof. We have $p(x) = \alpha\prod_\beta(x-\beta)\prod_\gamma(\gamma-x)\prod_\lambda(x-\lambda)(x-\overline{\lambda})$ where β are negative roots, γ are positive roots, λ are complex roots with positive imaginary parts, and $\alpha > 0$. Now

$$x - \beta = (x+1) + (-\beta - 1) \quad \text{and} \quad -\beta - 1 > 0\ .$$

Thus, $x - \beta \in B_1[-1,1]$ for all β. Similarly, $\gamma - x \in B_1[-1,1]$ for all γ. Now,

$(x-\lambda)(x-\overline{\lambda}) = x^2 - 2Re(\lambda x) + |\lambda|^2$ is a convex combination of $(|\lambda| - x)^2, x^2 + |\lambda|^2$;

$|\lambda| - x \in B_1[-1,1]$ implies $(|\lambda| - x)^2 \in B_2[-1,1]$;

$$x^2 + |\lambda|^2 = x^2 + 1 + |\lambda|^2 - 1 = \frac{1}{2}(x+1)^2 + \frac{1}{2}(x-1)^2 + |\lambda|^2 - 1 \in B_2[-1,1]\ .$$

∎

Lemma 27. *The Taylor polynomial P_{2n} of degree $2n$ of the function e^{-x^2} at 0 has no roots in the disc $\{|z| \leq \frac{\sqrt{n}}{\mathrm{e}}\}$.*

Proof. Let $|z| \leq \frac{\sqrt{n}}{\mathrm{e}}$. Then

$$|\mathrm{e}^{-z^2} - P_{2n}(z)| \leq \sum_{k>n} \frac{|z|^{2k}}{k!} \leq \sum_{k>n} (\frac{\mathrm{e}|z|^2}{k})^k \leq \sum_{k>n} (\frac{\mathrm{e}|z|^2}{n})^k \leq \sum_{k>n} \mathrm{e}^{-k} < \mathrm{e}^{-n} \leq |\mathrm{e}^{-z^2}| \ ,$$

and the result follows. $\square$

In the sequel, we will make use of the inequality

$$|\mathrm{e}^{-z^2} - P_{2n}(z)| < \mathrm{e}^{-n} \text{ for } |z| \leq \frac{\sqrt{n}}{\mathrm{e}}$$

obtained in the course of the last proof.

Lemma 28. *Suppose that ν is a positive measure on $\mathbb{R}$ such that $g = \nu * \mathrm{e}^{-nx^2}$ is bounded on the entire real line. Then there exists $p_n \in B_{200n}[-1,1]$ such that $\|g - p_n\|_{L^\infty[-1,1]} \leq 3\mathrm{e}^{-n}\|g\|_\infty$.*

Proof. Note that $x \mapsto P_{200n}(\sqrt{n}(x-t))$ has no roots in the disc $\{|z-t| \leq \frac{10}{\mathrm{e}}\}$. If $|t| \leq 2$, this disc contains the disc $\{|z| \leq 1\}$, so $P_{200n}(\sqrt{n}(x-t)) \in B_{200n}[-1,1]$. Now put

$$p_n = \nu[-2,2] * P_{200n}(\sqrt{n}\ \cdot) \in B_{200n}[-1,1] \ .$$

Clearly, for all $x \in [-1,1]$, we have

$$\begin{aligned} |g(x) - p_n(x)| &= \int_2^\infty \mathrm{e}^{-n(x-t)^2}\ d\nu(t) + \int_{-\infty}^{-2} \mathrm{e}^{-n(x-t)^2}\ d\nu(t) \\ &+ \int_{-2}^{2} |\mathrm{e}^{-n(x-t)^2} - P_{200n}(\sqrt{n}(x-t))|\ d\nu(t) =: I_1 + I_2 + I_3 \ . \end{aligned}$$

But $|\mathrm{e}^{-n(x-t)^2} - P_{200n}(\sqrt{n}(x-t))| \leq \mathrm{e}^{-100n}$ as long as $|x-t| \leq \frac{10}{\mathrm{e}}$. So $I_3 \leq \mathrm{e}^{-100n}\nu[-2,2]$. On the other hand, since

$$\int_{-1}^{1} \mathrm{e}^{-nx^2}\ dx = \frac{1}{\sqrt{n}} \int_{-\sqrt{n}}^{\sqrt{n}} \mathrm{e}^{-x^2}\ dx \geq \frac{1}{2\sqrt{n}} \ ,$$

we conclude that

$$\|g\|_\infty \cdot 6 \geq \int_{-3}^{3} g(x)dx \geq \frac{1}{2\sqrt{n}}\nu[-2,2] \ .$$

So $I_3 \leq \mathrm{e}^{-100n} 12\sqrt{n}\|g\|_\infty \leq \mathrm{e}^{-95n}\|g\|_\infty$.

Now, since for every $y > 0, z > 1$, we have $\mathrm{e}^{-n(y+z)^2} \leq \mathrm{e}^{-ny^2}\mathrm{e}^{-n}$, we obtain

$$\begin{aligned} I_1 &= \int_2^\infty \mathrm{e}^{-n(t-x)^2}\ d\nu(t) = \int_2^\infty \mathrm{e}^{-n((t-2)+(2-x))^2}\ d\nu(t) \\ &\leq \int_2^\infty \mathrm{e}^{-n} \cdot \mathrm{e}^{-n(t-2)^2}\ d\nu(t) \leq \mathrm{e}^{-n} g(2) \leq \mathrm{e}^{-n}\|g\|_\infty \ , \end{aligned}$$

and, similarly, $I_2 \leq \mathrm{e}^{-n}\|g\|_\infty$. Bringing these three estimates together, we arrive at the conclusion of the lemma. $\square$

Corollary 29. *Let* $E_n = \{\nu * \mathrm{e}^{-nx^2}\}$. *If* $f : \mathbb{R} \to [0,1]$ *can be approximated by the functions* $g_n \in E_n$ *on the entire line with an error* $O(n^{-\alpha/2})$, *then* f *can be approximated by* $p_n \in B_{200n}[-1,1]$ *with an error* $O(n^{-\alpha/2})$ *on* $[-1,1]$.

Proof. Obvious from Lemma 28. □

Now fix $\alpha \in (0,1)$. Our next task will be to construct a function $f : \mathbb{R} \to [0,1]$ that is approximable by functions $g_n \in E_{\pi n}$ with an error $O(n^{-\alpha/2})$ but not in the class $C^\alpha[-1/2,1/2]$. Note that $E_\lambda \subset E_{\lambda'}$ whenever $\lambda < \lambda'$, so it does not matter whether we consider only integer values or all real values of n in our statement.

Fix $h \in (0,1)$ and $m \in \mathbb{N}$ and define

$$f_{h,m} := h \sum_{k \in \mathbb{Z}} e^{-\pi m (x - kh/\sqrt{m})^2} = h \sum_{k \in \mathbb{Z}} e^{-\pi h^2 (k - x\sqrt{m}/h)^2}.$$

Recalling that the Fourier transform of the function $x \mapsto h e^{-\pi h^2 x^2}$ is $y \mapsto e^{-\pi y^2/h^2}$ and using the Poisson summation formula

$$\sum_{k \in \mathbb{Z}} F(k+x) = \sum_{\ell \in \mathbb{Z}} \widehat{F}(\ell) e^{2\pi i x},$$

we get

$$f_{h,m} = \sum_{\ell \in \mathbb{Z}} e^{-\pi \ell^2/h^2} e^{-2\pi i x \sqrt{m}/h}.$$

This representation immediately implies that $f_{h,m}$ attains its maximum at $x = 0$, and that

$$\begin{aligned} |f_{h,m} - 1| &\le \sum_{\ell \in \mathbb{Z} \setminus \{0\}} e^{-\pi \ell^2/h^2} \le 4 e^{-\pi/h^2} \\ f_{h,m}(0) - f_{h,m}\left(\frac{h}{2\sqrt{m}}\right) &= 2 \sum_{\ell \text{ odd}} e^{-\pi \ell^2/h^2} \ge 4 e^{-\pi/h^2}. \end{aligned}$$

Also note that $f_{h,m} \in E_{\pi m}$.

Now let Λ denote the set $\{2^j : j = 2, 3, 4, \ldots\}$. Choose h_m so that

$$\frac{e^{-\pi/h_m^2}}{(\log_2 m)^2} = \frac{1}{m^{\alpha/2}}.$$

This choice makes sense for $m \ge m_0(\alpha) \ge 4$. Define $\Lambda' := \{m \in \Lambda : m \ge m_0(\alpha)\}$ and

$$f := \sum_{m \in \Lambda'} \frac{1}{(\log_2 m)^2} f_{h_m,m}.$$

For every $n \in \mathbb{N}$, let

$$g_n := \sum_{m \in \Lambda', m \le n} \frac{1}{(\log_2 m)^2} f_{h,m} + \sum_{m \in \Lambda', m \le n} \frac{1}{(\log_2 m)^2} \in E_{\pi n}.$$

The second sum is just a constant. Now,

$$\begin{aligned} |f - g_n| &\le \sum_{m \in \Lambda', m > n} \frac{1}{(\log_2 m)^2} \|f_{h_m,m} - 1\|_\infty \le \sum_{m \in \Lambda', m > n} \frac{1}{(\log_2 m)^2} 4 e^{-\pi/h_m^2} \\ &= 4 \sum_{m \in \Lambda', m > n} m^{-\alpha/2} \le \operatorname{const} n^{-\alpha/2}. \end{aligned}$$

On the other hand, for every $m \in \Lambda'$, we have

$$f(0) - f\left(\frac{h_m}{2\sqrt{m}}\right) \geq \left(2 \sum_{\ell \text{ odd}} e^{-\pi\ell^2/h_m}\right) / (\log_2 m)^2 > 4\frac{e^{-\pi/h_m^2}}{(\log_2 m)^2} = 4m^{-\alpha/2}.$$

Thus,

$$\|f\|_{C^\alpha[-1/2,1/2]} \geq \frac{4m^{-\alpha/2}}{(h_m/2\sqrt{m})^\alpha} = 16h_m^{-\alpha} \to \infty \quad \text{as } m \to \infty,$$

so f is not in the class $C^\alpha[-1/2,1/2]$. So, we have obtained a function $f \notin C^\alpha[-1/2,1/2]$ such that it can be approximated by polynomials $p_n \in B_n[-1,1]$ at the rate $O(n^{-\alpha/2})$.

Consider the function $\widetilde{f}(x) := f(x) \cdot x(1-x)$ and the polynomials $\widetilde{p}_n(x) := p_n(x) \cdot x(1-x)$. The polynomials are in $B_{n+2}[-1,1]$ and the the function $\widetilde{f}$ satisfies the condition

$$|\widetilde{f}(x) - \widetilde{p}_n(x)| \leq x(1-x) \operatorname{const} n^{-\alpha/2} \leq \operatorname{const} \left(\sqrt{\frac{x(1-x)}{n}}\right)^\alpha \leq \operatorname{const} \Delta_n^\alpha(x).$$

Claim 10 is thus disproved.

7 Further questions and remarks

In this section we will make a few additional remarks on this and some related problems. We begin by discussing a conjectural characterization of simulation rates in case α is an integer. In that case, the classical problem of approximation a given function by polynomials of degree at most n already has a somewhat different solution, as we now explain.

Definition 30. *Let $\alpha \in \mathbb{N}$. A function f is said to be in the* ***smoothness class*** *$C^{\alpha*}[0,1]$ if f is $r := \lceil\alpha\rceil - 1$ times differentiable and the following condition holds:*

The ***symmetric modulus of continuity*** *of $f^{(r)}$*

$$\omega^*(f^{(r)}, h) := \sup_{t<h,\ x\in[t,1-t]} |f^{(r)}(x+t) - 2f^{(r)}(x) + f^{(r)}(x-t)|$$

is of order $O(h)$. In that case, we use the notation

$$\|f\|_{C^{\alpha*}} := \sup_h \frac{\omega^*(f^{(r)}, h)}{h}.$$

Remark 31. *The class $C^{\alpha*}[0,1]$ is also known as the* ***generalized Lipschitz class,*** *and, for $\alpha = 1$, as the* ***Zygmund class*** *(see, e.g., [1, Chap.2, Sec.9]).*

The characterization of the best polynomial approximation in case $\alpha \in \mathbb{N}$ is then given by the following result.

Result 32 (see, e.g., [1, Chapter 8, Theorem 6.3]). *Let $\alpha \in \mathbb{N}$. The best approximant p_n to f from the space Π_n of polynomials of degree at most n satisfies*

$$|p_n(x) - f(x)| = O(\Delta_n^{2\alpha}(x)) \qquad \text{for all } x \in [0,1]$$

if and only if $f \in C^{\alpha*}[0,1]$.

Motivated by this result on unrestricted polynomial approximation, we therefore conjecture a corresponding characterization of simulation rates.

Conjecture 33. *Let* $\alpha \in \mathbb{N}$. *Let* $f \in C^{\alpha *}[0,1]$ *be a function bounded strictly between* 0 *and* 1. *Then* f *can be simulated at the rate* $\Delta_n^{\alpha}(x)$ *on* $[0,1]$. *Precisely, there exist polynomials* g_n *and* f_n *satisfying conditions (i) – (iv) of Result 3 and bound (6).*

We can verify the converse (if f is simulable at the rate $\Delta_n^{\alpha}(x)$ on the interval $[0,1]$ where $\alpha \in \mathbb{N}$, then $f \in C^{\alpha *}[0,1]$) by adapting the argument of Section 3.

We now discuss a likely connection between the Besov smoothness of and the ability to simulate f so that the simulation time has a certain finite moment.

Definition 34. *The simulation time is said to* ***have finite*** α ***moment*** *if*

$$\sum_{n=1}^{\infty} n^{\alpha-1} P_p(N > n) = O(1).$$

If such a simulation algorithm of f *exists, then* f *is called* ***simulable with finite*** α ***moment****.*

Recall how the Besov spaces are defined (see, e.g., [1, pp.54–57]):

Definition 35. *Let* $\alpha > 0$ *be given and let* $r =: \lfloor \alpha \rfloor + 1$. *The* ***Besov space*** $B_1^{\alpha}(L_\infty(D))$ *is the collection of functions* $f \in L_\infty(D)$ *such that*

$$\begin{aligned} |f|_{B_1^{\alpha}(L_\infty)} &:= \|\omega_r(f;\cdot)\|_{\alpha,1} = \int_0^1 t^{-\alpha}\omega_r(f,t)\,\frac{dt}{t} < \infty, \quad \text{where} \\ \omega_r(f;t) &:= \sup_{0<h\le t} \|\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j} f(\cdot + jh)\|_\infty, \qquad t \ge 0. \end{aligned}$$

Conjecture 36. *Suppose that* $\alpha \notin \mathbb{N}$. *A function* $f : [0,1] \to [0,1]$ *can be simulated with a finite* α *moment if and only if* $f \in B_1^{\alpha}(L_\infty([0,1]))$.

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