

Not all GKK τ -matrices are stable

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Abstract

Hermitian positive definite, totally positive, and nonsingular M -matrices enjoy many common properties, in particular

- (A) positivity of all principal minors,
- (B) weak sign symmetry,
- (C) eigenvalue monotonicity,
- (D) positive stability.

The class of GKK matrices is defined by properties (A) and (B), whereas the class of nonsingular τ -matrices by (A) and (C). It was conjectured that

- (A), (B) \implies (D) [D. Carlson, J. Res. Nat. Bur. Standards Sect. B 78 (1974) 1-2],
- (A), (C) \implies (D) [G.M. Engel and H. Schneider, Linear and Multilinear Algebra 4 (1976) 155-176],
- (A), (B) \implies a property stronger than (D) [R. Varga, Numerical Methods in Linear Algebra, 1978, pp.5-15],
- (A), (B), (C) \implies (D) [Hershkowitz, Linear Algebra Appl. 171 (1992) 161-186].

We describe a class of unstable GKK τ -matrices, thus disproving all four conjectures.

1 Definitions and notation

Given a matrix $A \in \mathbb{C}^{n \times n}$, let $A(\alpha, \beta)$ denote the submatrix of A whose rows are indexed by α and columns by β ($\alpha, \beta \in \langle n \rangle := \{1, \dots, n\}$) and let $A[\alpha, \beta]$ denote $\det A(\alpha, \beta)$ if $\#\alpha = \#\beta$ (where $\#$ stands for the cardinality of a set) with the convention $A[\emptyset, \emptyset] := 1$.

A matrix A is called a P -matrix if $A[\alpha, \alpha] > 0 \forall \alpha \subseteq \langle n \rangle$. A is *weakly sign-symmetric* if

$$A[\alpha, \beta]A[\beta, \alpha] \geq 0 \quad \forall \alpha, \beta \in \langle n \rangle, \quad \#\alpha = \#\beta = \#\alpha \cup \beta - 1.$$

Weakly sign-symmetric P -matrices are also called *GKK* after Gantmacher, Krein, and Kotelyansky. It was proven by Gantmacher, Krein [5], and Carlson [2] that a P -matrix is GKK iff it satisfies the generalized Hadamard-Fisher inequality

$$A[\alpha, \alpha]A[\beta, \beta] \geq A[\alpha \cup \beta, \alpha \cup \beta]A[\alpha \cap \beta, \alpha \cap \beta] \quad \forall \alpha, \beta \subseteq \langle n \rangle. \quad (1)$$

Carlson [3] conjectured that the GKK matrices are positive stable, i.e., $\operatorname{Re} \lambda > 0 \forall \lambda \in \sigma(A)$ (here $\sigma(A)$ denotes, as usual, the spectrum of A), and showed his conjecture to be true for $n \leq 4$.

Let

$$l(A) := \begin{cases} \min\{\lambda \in \sigma(A) \cap \mathbb{R}\} & \text{if } \sigma(A) \cap \mathbb{R} \neq \emptyset \\ \infty & \text{otherwise.} \end{cases}$$

A is called an ω -matrix if it has eigenvalue monotonicity

$$l(A(\alpha, \alpha)) \leq l(A(\beta, \beta)) < \infty \quad \text{whenever } \emptyset \neq \beta \subseteq \alpha \subseteq \langle n \rangle.$$

A is a τ -matrix if, in addition, $l(A) \geq 0$.

Engel and Schneider [4] asked if nonsingular τ -matrices or, equivalently, ω -matrices all whose principal minors are positive (see Remark 3.7 in [4]), are positive stable. Varga [9] conjectured even more than stability, viz.

$$|\arg(\lambda - l(A))| \leq \frac{\pi}{2} - \frac{\pi}{n} \quad \forall \lambda \in \sigma(A).$$

This inequality was proven for $n \leq 3$ by Varga (unpublished) and Hershkowitz and Berman [7] and for $n = 4$ by Mehrmann [8]. In his survey paper [6], Hershkowitz posed the weaker conjecture that τ -matrices that are also GKK are stable.

Below we describe a class of GKK τ -matrices which are not even nonnegative stable, i.e., have eigenvalues with negative real part. We construct Toeplitz Hessenberg matrices $A_{n,k,t}$ of size n for $k \in \mathbb{N}$ and $t \in \mathbb{R}$. We show that $A_{n,k,t}$ is GKK for any $t \in (0, 1)$, a τ -matrix if $n \leq 2k+2$ and $t \in (0, 1)$ is sufficiently small, and that $A_{2k+2,k,t}$ is unstable for sufficiently large k and sufficiently small positive t . This provides a counterexample to the Hershkowitz conjecture and, therefore, to the Carlson, Engel and Schneider, and Varga conjectures as well.

In what follows, we shall use the following notation

$$p:q := \begin{cases} \{p, p+1, \dots, q\} & \text{if } p \leq q \\ \emptyset & \text{otherwise} \end{cases} \quad \forall p, q \in \mathbb{N}, \quad x_+ := \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad \forall x \in \mathbb{R}.$$

2 Counterexample

Given $k, n \in \mathbb{N}$, and $t \in (0, 1)$, let $A_{n,k,t}$ be the following Toeplitz Hessenberg matrix. If $n \leq k+1$, set

$$A_{n,k,t} := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 1 \end{pmatrix}_{n \times n}.$$

Otherwise let

$$A_{n,k,t} := \begin{pmatrix} 1 & \overbrace{0 \cdots 0}^k & 0 & 0 & a_1^{k,t} & a_2^{k,t} & \cdots & a_{n-k-2}^{k,t} & a_{n-k-1}^{k,t} \\ 1 & 1 & \cdots & 0 & 0 & a_1^{k,t} & \cdots & a_{n-k-3}^{k,t} & a_{n-k-2}^{k,t} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & a_1^{k,t} & a_2^{k,t} \\ 0 & 0 & \cdots & 1 & 1 & 0 & \cdots & 0 & a_1^{k,t} \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}_{n \times n}$$

where $a_j^{k,t}$'s are chosen so that $A_{n,k,t}[\langle k+j+1 \rangle, \langle k+j+1 \rangle] = t^j$. This definition makes sense for all $j = 1, \dots, n-k-1$. Indeed, the expansion of $A_{n,k,t}[\langle k+j+1 \rangle, \langle k+j+1 \rangle]$ by the first row gives

$$\begin{aligned} A_{n,k,t}[\langle k+j+1 \rangle, \langle k+j+1 \rangle] & (= \det A_{k+j+1,k,t}) = \\ & A_{n,k,t}[2:k+j+1, 2:k+j+1] + \sum_{l=1}^j (-1)^{k+l} a_l^{k,t} A_{n,k,t}[k+l+2:k+j+1, k+l+2:k+j+1] = \\ & \det A_{k+j,k,t} + \sum_{l=1}^j (-1)^{k+l} a_l^{k,t} \det A_{j-l,k,t} \end{aligned} \quad (2)$$

(recall that $A_{n,k,t}[\emptyset, \emptyset] = 1$, so the last term in the sum is well defined). As the coefficient of $a_j^{k,t}$ in Eq. (2) is equal to $(-1)^{k+j}$, the equation $A_{n,k,t} = s$ (linear in $a_j^{k,t}$) has a solution for any right hand side s , in particular, for $s = t^j$. Since $A_{n,k,t}$ is Toeplitz, this implies $A_{n,k,t}[i:i+j-1, i:i+j-1] = t^{(j-k-1)_+}$.

Show that the matrices $A_{n,k,t}$ are GKK for any $t \in (0, 1)$. Since $A_{n,k,t}$ is Hessenberg, the submatrix $A_{n,k,t}(\langle n \rangle \setminus i:i+j-1, \langle n \rangle \setminus i:i+j-1)$ is block upper triangular if $1 < i \leq i+j-1 < n$, so

$$A_{n,k,t}[\alpha \cup \beta, \alpha \cup \beta] = A_{n,k,t}[\alpha, \alpha]A_{n,k,t}[\beta, \beta] \quad \text{whenever} \quad i < j - 1 \text{ for all } i \in \alpha, j \in \beta. \quad (3)$$

This shows that $A_{n,k,t}$ is a P -matrix. Moreover, since $0 < t < 1$ and

$$(x + y - k - 1)_+ + (x + z - k - 1)_+ \leq (x - k - 1)_+ + (x + y + z - k - 1)_+ \quad \forall x, y, z \geq 0,$$

we have

$$\begin{aligned} & A_{n,k,t}[i:i+j-1, i:i+j-1] \cdot A_{n,k,t}[l:l+m-1, l:l+m-1] = \\ & \quad t^{(j-k-1)_++(m-k-1)_+} \geq t^{(l+m-i-k-1)_++(i+j-l-k-1)_+} = \quad \text{if } l \leq i+j-1. \quad (4) \\ & A_{n,k,t}[i:l+m-1, i:l+m-1] \cdot A_{n,k,t}[l:i+j-1, l:i+j-1] \end{aligned}$$

Together with Eq. (3), Eq. (4) shows that $A_{n,k,t}$ satisfies Eq. (1) if α, β are sets of consecutive integers.

To prove Eq. (1) in general, first make a definition. Call the subsets $\alpha, \beta \subseteq \langle n \rangle$ *separated* if $|p - q| > 1 \forall p \in \alpha, q \in \beta$. Suppose $\alpha, \beta_1, \dots, \beta_j \subseteq \langle n \rangle$ are sets of consecutive integers, β_i ($i = 1, \dots, j$) are separated, and

$$\text{for any } i = 1, \dots, j, \text{ there exist } p \in \beta_i \text{ and } q \in \alpha \text{ such that } |p - q| \leq 1. \quad (5)$$

Then $A_{n,k,t}$, α , and $\beta := \cup_{i=1}^j \beta_i$ satisfy Eq. (1). Indeed, Eq. (1) holds for α and β_1 . If $1 \leq l < j$, then, assuming Eq. (1) for α and $\gamma_l := \cup_{i=1}^l \beta_i$, we have

$$\begin{aligned} & A_{n,k,t}[\alpha, \alpha]A_{n,k,t}[\gamma_{l+1}, \gamma_{l+1}] = A_{n,k,t}[\alpha, \alpha]A_{n,k,t}[\gamma_l, \gamma_l]A_{n,k,t}[\beta_{l+1}, \beta_{l+1}] \geq \\ & \quad A_{n,k,t}[\alpha \cup \gamma_l, \alpha \cup \gamma_l]A_{n,k,t}[\alpha \cap \gamma_l, \alpha \cap \gamma_l]A_{n,k,t}[\beta_{l+1}, \beta_{l+1}]. \end{aligned}$$

Due to Eq. (5), $\alpha \cup \gamma_l$ is a set of consecutive integers, so an application of Eq. (1) yields

$$A_{n,k,t}[\alpha \cup \gamma_l, \alpha \cup \gamma_l]A_{n,k,t}[\beta_{l+1}, \beta_{l+1}] \geq A_{n,k,t}[\alpha \cup \gamma_{l+1}, \alpha \cup \gamma_{l+1}]A_{n,k,t}[(\alpha \cup \gamma_l) \cap \beta_{l+1}, (\alpha \cup \gamma_l) \cap \beta_{l+1}].$$

But $(\alpha \cup \gamma_l) \cap \beta_{l+1} = \alpha \cap \beta_{l+1}$ since the sets β_i are pairwise disjoint. So,

$$\begin{aligned} & A_{n,k,t}[\alpha, \alpha]A_{n,k,t}[\gamma_{l+1}, \gamma_{l+1}] \geq A_{n,k,t}[\alpha \cup \gamma_{l+1}, \alpha \cup \gamma_{l+1}]A_{n,k,t}[\alpha \cap \gamma_l, \alpha \cap \gamma_l]A_{n,k,t}[\alpha \cap \beta_{l+1}, \alpha \cap \beta_{l+1}] = \\ & \quad A_{n,k,t}[\alpha \cup \gamma_{l+1}, \alpha \cup \gamma_{l+1}]A_{n,k,t}[\alpha \cap \gamma_{l+1}, \alpha \cap \gamma_{l+1}]. \quad (6) \end{aligned}$$

Now, given a set of consecutive integers $\alpha \subseteq \langle n \rangle$ and any index set $\beta \subseteq \langle n \rangle$, write $\beta = \gamma_1 \cup \gamma_2$ where $\gamma_1 := \cup_{i=1}^l \beta_i$, $\gamma_2 := \cup_{i=l+1}^{l+m} \beta_i$, all β_i ($i = 1, \dots, l+m$) are separated, and β_i satisfies Eq. (5) if and only if $i \leq l$. Then

$$\begin{aligned} & A_{n,k,t}[\alpha, \alpha]A_{n,k,t}[\beta, \beta] = A_{n,k,t}[\alpha, \alpha]A_{n,k,t}[\gamma_1, \gamma_1]A_{n,k,t}[\gamma_2, \gamma_2] \geq \\ & \quad A_{n,k,t}[\alpha \cup \gamma_1, \alpha \cup \gamma_1]A_{n,k,t}[\alpha \cap \gamma_1, \alpha \cap \gamma_1]A_{n,k,t}[\gamma_2, \gamma_2] = \\ & \quad A_{n,k,t}[\alpha \cup \gamma_1 \cup \gamma_2, \alpha \cup \gamma_1 \cup \gamma_2]A_{n,k,t}[\alpha \cap \gamma_1, \alpha \cap \gamma_1] = A_{n,k,t}[\alpha \cup \beta, \alpha \cup \beta]A_{n,k,t}[\alpha \cap \beta, \alpha \cap \beta]. \end{aligned}$$

In other words, $A_{n,k,t}$ satisfies Eq. (1) if $\alpha \subseteq \langle n \rangle$ is a set of consecutive integers and $\beta \subseteq \langle n \rangle$ is arbitrary.

Finally, if $\alpha_1, \alpha_2, \beta \subseteq \langle n \rangle$, the sets α_i ($i = 1, 2$) are separated, Eq. (1) holds for α_1 and β , and α_2 is a set of consecutive integers, then Eq. (1) holds for $\alpha := \alpha_1 \cup \alpha_2$ and β :

$$\begin{aligned} & A_{n,k,t}[\alpha, \alpha]A_{n,k,t}[\beta, \beta] = A_{n,k,t}[\alpha_1, \alpha_1]A_{n,k,t}[\alpha_2, \alpha_2]A_{n,k,t}[\beta, \beta] \geq \\ & \quad A_{n,k,t}[\alpha_1 \cup \beta, \alpha_1 \cup \beta]A_{n,k,t}[\alpha_1 \cap \beta, \alpha_1 \cap \beta]A_{n,k,t}[\alpha_2, \alpha_2] \geq \\ & \quad A_{n,k,t}[(\alpha_1 \cup \beta) \cup \alpha_2, (\alpha_1 \cup \beta) \cup \alpha_2]A_{n,k,t}[(\alpha_1 \cup \beta) \cap \alpha_2, (\alpha_1 \cup \beta) \cap \alpha_2]A_{n,k,t}[\alpha_1 \cap \beta, \alpha_1 \cap \beta] = \\ & \quad A_{n,k,t}[\alpha \cup \beta, \alpha \cup \beta]A_{n,k,t}[\alpha_1 \cap \beta, \alpha_1 \cap \beta]A_{n,k,t}[\alpha_2 \cap \beta, \alpha_2 \cap \beta] = \\ & \quad A_{n,k,t}[\alpha \cup \beta, \alpha \cup \beta]A_{n,k,t}[\alpha \cap \beta, \alpha \cap \beta]. \end{aligned}$$

So, by induction on the number of 'components' of α , Eq. (1) holds for any $\alpha, \beta \subseteq \langle n \rangle$. Thus, by the Gantmacher-Krein-Carlson theorem, $A_{n,k,t}$ is GKK for any $t \in (0, 1)$ and any $k, n \in \mathbb{N}$.

Now check that $A_{n,k,t}$ have eigenvalue monotonicity if $n \leq 2k + 2$ and $t \in (0, 1)$ is sufficiently small. Let $\varphi_j^{k,t}(\lambda) := \det(A_{k+j+1,k,t} - \lambda I)$ for $j = 1, \dots, k + 1$. Show by induction that

$$\varphi_j^{k,t}(\lambda) = \begin{cases} (1-\lambda)^{k+2} - (1-t) & \text{if } j = 1 \\ (1-\lambda)^{j+k+1} - j(1-t)(1-\lambda)^{j-1} + (j-1)(1-t)^2(1-\lambda)^{j-2} + \\ \frac{t(1-t)^2}{((1-\lambda)-t)^2} [t^{j-1} - (j-1)t(1-\lambda)^{j-2} + (j-2)(1-\lambda)^{j-1}] & \text{if } j > 1, \end{cases} \quad (7)$$

$$a_j^{k,t} = \begin{cases} (-1)^k(1-t) & \text{if } j = 1 \\ (-1)^{k+j}t^{j-2}(1-t)^2 & \text{if } j > 1, \end{cases} \quad (8)$$

$$g_j^{k,t}(\lambda) := (-1)^{k+1} \det \begin{pmatrix} (1-\lambda) & 0 & \dots & 0 & a_j^{k,t} \\ 1 & (1-\lambda) & \dots & 0 & a_{j-1}^{k,t} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & (1-\lambda) & a_2^{k,t} \\ 0 & 0 & \dots & 1 & a_1^{k,t} \end{pmatrix} = \\ - (1-t)(1-\lambda)^{j-1} + (1-t)^2 \frac{(1-\lambda)^{j-1} - t^{j-1}}{(1-\lambda) - t} \quad \forall j \in \mathbb{N}. \quad (9)$$

By direct calculation, $\varphi_1^{k,t}(\lambda) = (1-\lambda)^{k+2} - (-1)^k a_1^{k,t}$, so, since $\varphi_1^{k,t}(0) = t$, we have $a_1^{k,t} = (-1)^k(1-t)$. Thus Eqs. (7)–(9) hold for $j = 1$. Now suppose that $j \geq 2$ and our formulas are true for $j - 1$. Expansion of $\varphi_j(\lambda)$ by its last row gives

$$\varphi_j^{k,t}(\lambda) = (1-\lambda)\varphi_{j-1}^{k,t}(\lambda) + g_j^{k,t}(\lambda). \quad (10)$$

Since $\varphi_j^{k,t}(0) = t^j \forall j \in \mathbb{N}$, this implies $g_j^{k,t}(0) = t^j - t^{j-1}$. On the other hand, expanding $g_j^{k,t}(\lambda)$ by its first row, we get

$$g_j^{k,t}(\lambda) = (1-\lambda)g_{j-1}^{k,t}(\lambda) + (-1)^{j+k}a_j^{k,t}, \quad (11)$$

so $a_j^{k,t} = (-1)^{k+j}[g_j^{k,t}(\lambda) - (1-\lambda)g_{j-1}^{k,t}(\lambda)]_{\lambda=0} = (-1)^{j+k}t^{j-2}(1-t)^2$, which gives Eq. (8). Now, using Eq. (11) again together with the inductive hypothesis on $g_{j-1}^{k,t}$, we get Eq. (9):

$$g_j^{k,t}(\lambda) = - (1-t)(1-\lambda)^{j-1} + (1-t)^2 \frac{(1-\lambda)^{j-1} - (1-\lambda)t^{j-2}}{(1-\lambda) - t} + t^{j-2}(1-t)^2 = \\ - (1-t)(1-\lambda)^{j-1} + (1-t)^2 \frac{(1-\lambda)^{j-1} - t^{j-1}}{(1-\lambda) - t}.$$

Finally, substituting the expression for $\varphi_{j-1}^{k,t}(\lambda)$ and the just verified expression for $g_j^{k,t}(\lambda)$ into Eq. (10) yields Eq. (7).

If $\langle n \rangle \supseteq \alpha = \cup_{i=1}^j \alpha_i$ is the union of separated sets of consecutive integers, then $\det(A_{n,k,t}(\alpha, \alpha) - \lambda I) = \prod_{i=1}^j \det(A(\alpha_i, \alpha_i) - \lambda I)$ since $A_{n,k,t} - \lambda I$ is Hessenberg (the same observation earlier led to Eq. (3)). Since $A_{n,k,t} - \lambda I$ is Toeplitz, the product in the right hand side equals $\prod_{i=1}^j \det(A_{n,k,t}(\langle \# \alpha_i \rangle, \langle \# \alpha_i \rangle) - \lambda I)$. Hence, to prove eigenvalue monotonicity of $A_{n,k,t}$ for $n \leq 2k + 2$ it is enough to prove it for leading principal submatrices of $A_{n,k,t}$ only, i.e., to show

$$l(A_{k+j+1,k,t}) \leq l(A_{k+j,k,t}) \quad \forall j \in \mathbb{N},$$

i.e., that $\varphi_j^{k,t}$ has a root in $(0, 1]$ for any $j \leq k + 1$, and

$$\min\{\lambda \in (0, 1] : \varphi_j(\lambda) = 0\} \leq \min\{\lambda \in (0, 1] : \varphi_{j-1}(\lambda) = 0\}, \quad j = 2, \dots, k + 1$$

(since $A_{k+j,k,t}$ is a P -matrix, the coefficients of its characteristic polynomial are strictly alternating, so $A_{k+j,k,t}$ has no nonpositive eigenvalues). Observe that $\varphi_j^{k,t}(\lambda) = t^j - \lambda \tilde{\varphi}_j^{k,t}(\lambda)$ where

$$\tilde{\varphi}_j^{k,t}(\lambda) = - \frac{d\varphi_j^{k,t}(\lambda)}{d\lambda} \Big|_{\lambda=0} \xrightarrow{t \rightarrow 0+} = - \frac{d\nu_j^k(\lambda)}{d\lambda} \Big|_{\lambda=0},$$

$$\nu_j^k(\lambda) := \lim_{t \rightarrow 0^+} \varphi_j^{k,t}(\lambda) = (1-\lambda)^{j+k+1} - j(1-\lambda)^{j-1} + (j-1)(1-\lambda)^{j-2}.$$

So, $\lim_{t \rightarrow 0^+} \tilde{\varphi}_j^{k,t}(0) = k+3-j \geq 2 \quad \forall j = 1, \dots, k+1$.

Since 0, the minimal real root of ν_j^k , is simple, the minimal real root λ_j of $\varphi_j^{k,t}$ is positive and simple for all $j = 1, \dots, k+1$ whenever t is sufficiently small. But then $\tilde{\varphi}_j^{k,t}(\lambda_j)$ is bounded below by a positive constant for any $j = 1, \dots, k+1$, hence

$$\lambda_j = \frac{t^j}{\tilde{\varphi}_j^{k,t}(\lambda_j)} < \frac{t^{j-1}}{\tilde{\varphi}_{j-1}^{k,t}(\lambda_{j-1})} = \lambda_{j-1} \quad \forall j = 1, \dots, k+1$$

if t is small. So, for any $k \in \mathbb{N}$ and $n \leq 2k+2$, there exists $t(k) \in (0, 1)$ such that $A_{n,k,t}$ is a τ -matrix for all $t \in (0, t(k))$.

Now let $B_k := \lim_{t \rightarrow 0^+} A_{2k+2,k,t}$. The matrix B_k is Toeplitz with first column

$$(1, 1, \underbrace{0, \dots, 0}_{2k \text{ times}})^T$$

and first row

$$(1, \underbrace{0, \dots, 0}_k, (-1)^k, (-1)^k, \underbrace{0, \dots, 0}_{k-1}).$$

Show that there exists $K \in \mathbb{N}$ such that, for all $k > K$, B_k has an eigenvalue λ with $\operatorname{Re} \lambda < 0$. As the eigenvalues depend continuously on the entries of the matrix, this will demonstrate that, for any $k > K$, there exists $t \in (0, 1)$ such that the GKK τ -matrix $A_{2k+2,k,t}$ has an eigenvalue with negative real part.

The polynomial ν_{k+1}^k has a root with negative real part iff the polynomial ψ_k where

$$\psi_k(\lambda) := \frac{\nu_{k+1}^k(-\lambda)}{(1+\lambda)^{k-1}} = (1+\lambda)^{k+3} - (k+1)(1+\lambda) + k$$

has a root with positive real part. Since

$$\psi_k(\lambda) = \lambda \left[\sum_{j=0}^{k+1} \binom{k+3}{j} \lambda^{k+3-j-1} + 2 \right],$$

it is, in turn, enough to show that η_k where

$$\eta_k(\lambda) := \lambda^{k+3} \psi_k \left(\frac{1}{\lambda} \right) = 2\lambda^{k+2} + \sum_{j=2}^{k+3} \binom{k+3}{j} \lambda^{k+3-j}$$

has a root with positive real part. The Hurwitz matrix for the polynomial η_k is

$$H_k := \begin{pmatrix} \binom{k+3}{2} & \binom{k+3}{4} & \binom{k+3}{6} & \binom{k+3}{8} & \binom{k+3}{10} & \dots \\ 2 & \binom{k+3}{3} & \binom{k+3}{5} & \binom{k+3}{7} & \binom{k+3}{9} & \dots \\ 0 & \binom{k+3}{2} & \binom{k+3}{4} & \binom{k+3}{6} & \binom{k+3}{8} & \dots \\ 0 & 2 & \binom{k+3}{3} & \binom{k+3}{5} & \binom{k+3}{7} & \dots \\ 0 & 0 & \binom{k+3}{2} & \binom{k+3}{4} & \binom{k+3}{6} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{(k+2) \times (k+2)}.$$

Compute the minor $H_k[2:5, 2:5]$, taking out the factors $\binom{k+3}{2}$, $\binom{k+3}{4}$, $\binom{k+3}{6}$ from its second, third, and fourth columns respectively. We obtain

$$H_k[2:5, 2:5] = -\frac{1}{132300} (3k^3 - 49k^2 - 210k - 318)(k+4)^2(k+5) \binom{k+3}{2} \binom{k+3}{4} \binom{k+3}{6}.$$

It follows that $H_k[2:5, 2:5] < 0$ for k large enough, precisely, for all $k > 20$. But the Hurwitz matrix of a nonpositive stable polynomial is totally nonnegative (see [1]). So, for $k > 20$, η_k has a zero with positive real part, therefore, ν_{k+1}^k has a zero with negative real part. This completes the proof that the GKK τ -matrices $A_{2k+2, k, t}$ are unstable for sufficiently large k and small t .

Remark To illustrate the result, consider the matrix $A_{44, 21, 1/2}$, i.e., the Toeplitz matrix whose first column is

$$(1, 1, \underbrace{0, \dots, 0}_{42 \text{ times}})^T$$

and first row is

$$(1, \underbrace{0, \dots, 0}_{21 \text{ times}}, -1/2, -1/2^2, 1/2^3, -1/2^4, \dots, -1/2^{22})$$

and the limit matrix B_{21} , with the same first column as $A_{44, 21, 1/2}$ and first row equal to

$$(1, \underbrace{0, \dots, 0}_{21 \text{ times}}, -1, -1, \underbrace{0, \dots, 0}_{20 \text{ times}}).$$

According to MATLAB, the two eigenvalues with minimal real part of the first matrix are

$$-2.809929189497896 \cdot 10^{-2} \pm 3.275076252367531 \cdot 10^{-1}i;$$

those of the second are

$$-3.420708309454068 \cdot 10^{-2} \pm 3.400425852703498 \cdot 10^{-1}i.$$

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