Hyperdeterminantal relations among symmetric principal minors

Olga Holtz
Mathematics Department
University of California
Berkeley, CA 94720

Bernd Sturmfels
Mathematics Department
University of California
Berkeley, CA 94720

January 12, 2007

Abstract

The principal minors of a symmetric $n{\times}n$-matrix form a vector of length 2^n. We characterize these vectors in terms of algebraic equations derived from the $2{\times}2{\times}2$-hyperdeterminant.

Key words. Hyperdeterminant, tensors, principal minors, symmetric matrices, minor assignment problem.

AMS subject classification. 14M12, 15A15, 15A72, 15A29, 13P10

1 Introduction

The principal minors of a real symmetric $n{\times}n$-matrix A form a vector of length 2^n. This vector is denoted A_*, and its entries are indexed by subsets I of $[n]:=\{1,2,\ldots,n\}$. Namely, A_I denotes the minor of A whose rows and columns are indexed by I. This includes the $0{\times}0$-minor $A_\emptyset = 1$. The aim of this paper is to give an algebraic characterization of all vectors in $\mathbb{R}^{2^n}$ which arise in this form. Our question can be rephrased as follows. We write $\mathbb{R}^{\binom{n+1}{2}}$ for the space of real symmetric $n{\times}n$-matrices. Our aim is to determine the image of the *principal minor map* $\phi : \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}^{2^n}$, $A \mapsto A_*$. If $n = 2$ then the image of ϕ is characterized by the trivial equation $A_\emptyset = 1$ and the one inequality

$$A_\emptyset \cdot A_{12} \quad \leq \quad A_1 \cdot A_2. \tag{1}$$

For $n \geq 2$, the entries of the symmetric matrix $A = (a_{ij})$ are determined up to sign by their principal minors of size $1{\times}1$ and $2{\times}2$, in view of the relation $a_{ij}^2 = A_iA_j - A_{ij}A_\emptyset$.

Remark 1. *The image of the principal minor map ϕ is a closed subset of $\mathbb{R}^{2^n}$. The same holds for the map $\phi_\mathbb{C} : \mathbb{C}^{\binom{n+1}{2}} \to \mathbb{C}^{2^n}$ which takes a complex symmetric matrix to its principal minors.*

Proof. Let A_* be a vector in $\mathbb{R}^{2^n}$ (resp. in $\mathbb{C}^{2^n}$) that lies in the closure of the image of ϕ (resp. the image of $\phi_\mathbb{C}$). Consider any sequence $\{A^{(k)}\}_{k\geq 0}$ of symmetric $n{\times}n$-matrices whose principal minors tend to A_* as $k \to \infty$. This sequence must be uniformly bounded, since the diagonal entries of $A^{(k)}$ have prescribed limits, and so do the magnitudes of all off-diagonal entries. By compactness, we can extract a convergent subsequence $A^{(k_j)}$, whose limit A therefore has principal minors A_*. $\square$

Remark 1 implies that the image of ϕ_C a complex algebraic variety in $\mathbb{C}^{2^n}$. That variety is the object of study in this paper. Equivalently, we seek to determine all polynomial equations that are valid on the image of the map $\phi : \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}^{2^n}$. Inequalities such as (1) will be disregarded.

To Bernd: "disregarded" is not quite right in view of our use of inequalities (12) later.

For $n = 3$, the matrix A has six distinct entries, so the seven non-trivial minors A_I must satisfy one polynomial equation. Expanding the determinant A_{123}, the desired equation is found to be

$$(A_{123} - A_{12}A_3 - A_{13}A_2 - A_{23}A_1 + 2A_1A_2A_3)^2 \quad = \quad 4 \cdot (A_1A_2 - A_{12})(A_2A_3 - A_{23})(A_1A_3 - A_{13}),$$

Our point of departure is the observation that this equation coincides with the *hyperdeterminant* of format 2×2×2. Namely, after homogenization using $A_\emptyset = 1$, this equation is equivalent to

$$\begin{aligned} &A_\emptyset^2 A_{123}^2 + A_1^2 A_{23}^2 + A_2^2 A_{13}^2 + A_3^2 A_{12}^2 + 4 \cdot A_\emptyset A_{12} A_{13} A_{23} + 4 \cdot A_1 A_2 A_3 A_{123} \\ &- 2 \cdot A_\emptyset A_1 A_{23} A_{123} - 2 \cdot A_\emptyset A_2 A_{13} A_{123} - 2 \cdot A_\emptyset A_3 A_{12} A_{123} - 2 \cdot A_1 A_2 A_{13} A_{23} \\ &- 2 \cdot A_1 A_3 A_{12} A_{23} - 2 \cdot A_2 A_3 A_{12} A_{13} \quad = \quad 0. \end{aligned} \tag{2}$$

Replacing subsets of $\{1, 2, 3\}$ by binary strings in $\{0, 1\}^3$, and thus setting $A_\emptyset = a_{000}, A_1 = a_{100}, \ldots, A_{123} = a_{111}$, we see that this polynomial has the full symmetry group of the 3-cube, and it does indeed coincide with the familiar formula for the hyperdeterminant (see [4, Prop. 14.B.1.7]). The *hyperdeterminantal relations* on A_* (of format 2×2×2) are obtained from (2) by substituting

$$\begin{aligned} &A_\emptyset \mapsto A_I \ , \quad A_{123} \mapsto A_{I\cup\{j_1,j_2,j_3\}} \ , \\ &A_1 \mapsto A_{I\cup\{j_1\}} \ , \ A_2 \mapsto A_{I\cup\{j_2\}} \ , \ A_3 \mapsto A_{I\cup\{j_3\}} \\ &A_{12} \mapsto A_{I\cup\{j_1,j_2\}} \ , \ A_{13} \mapsto A_{I\cup\{j_1,j_3\}} \ , \ A_{23} \mapsto A_{I\cup\{j_2,j_3\}} \end{aligned} \tag{3}$$

for any subset $I \subset [n]$ and any elements $j_1, j_2, j_3 \in [n]\backslash I$. Our first result states:

Theorem 2. *The principal minors of a symmetric matrix satisfy the hyperdeterminantal relations.*

The proof of this theorem is presented in Section 2, after reviewing some relevant background from matrix theory and the history of our problem. We also comment on connections to probability theory. A key question is whether the converse to Theorem 2 holds, i.e., whether every vector of length 2^n which satisfies the hyperdeterminantal relations arises from the principal minors of a (possibly complex) symmetric $n\times n$-matrix. The answer to this question is "not quite but almost". A useful counterexample to keep in mind is the following point on the hyperdeterminantal locus.

Example 3. For $n \geq 4$, define the vector $A_* \in \mathbb{R}^{2^n}$ by $A_\emptyset = 1$, $A_{123\cdots n} = -1$ and $A_I = 0$ for all other subsets I of $[n]$. Then A_* satisfies all hyperdeterminantal relations. It also satisfies all the *Hadamard-Fischer inequalities,* which are the generalization of the inequality (1) to arbitrary n:

$$A_{I\cap J} \cdot A_{I\cup J} \quad \leq \quad A_I \cdot A_J \qquad \text{for all} \quad I, J \subseteq [n]. \tag{4}$$

These inequalities hold for positive semidefinite symmetric matrices (see, e.g., [2]).

Nonetheless, the vector A_* is not the vector of principal minors of any symmetric $n\times n$-matrix. To see this, we note that a symmetric $n\times n$-matrix A which has all diagonal entries A_i and all principal 2×2-minors A_{ij} zero must be the zero matrix, so its determinant $A_{123\cdots n}$ would be zero.

We spell out different versions of a converse to Theorem 2 in the later sections. In Section 3 we derive a converse under a genericity hypothesis, and in Section 5 we discuss larger hyperdeterminants and derive a converse using so-called condensation relations. The ultimate converse would be an explicit list of generators for the prime ideal P_n of the algebraic variety image($\phi_{\mathbb{C}}$). In Section 6 we conjecture that the ideal P_n is generated by quartics, namely, the orbit of the hyperdeterminantal relations under a natural group action. Section 4 verifies this conjecture for $n = 4$. Theorem 2 together with these converses resolves the *Symmetric Principal Minor Assignment Problem* which was stated as an open question in Problem 3.4 of [9] and in Section 3.2 of [14].

2 Matrix Theory and Probability

This work is motivated by a number of recent results and problems from matrix theory and probability. Information about the principal minors of a given matrix is crucial in many matrix-theoretic settings. Of interest may be their exact value, their sign, or inequalities they satisfy. Among these problems are detection of P-matrices [13] and of GKK-matrices [8, 9], counting spanning trees of a graph [5] and the inverse multiplicative eigenvalue problem [3]. The *Principal Minor Assignment Problem,* as formulated in [9], is to determine whether a given vector A_* of length 2^n is realizable as the vector of all principal minors of some $n\times n$-matrix A. Very recently, Griffin and Tsatsomeros gave an algorithmic solution to this problem [6, 7]. Their work gives an algorithm, which, under a certain "genericity" condition, either outputs a solution matrix or determines that none exists. Our approach offers a more conceptual algebraic solution in the case of symmetric matrices.

In probability theory, information about principal minors is important in *determinantal point processes.* Determinantal processes arise naturally in several fields, including quantum mechanics of fermions [11], eigenvalues of random matrices, random spanning trees and nonintersecting paths (see [10] and references therein). A determinantal point process on a locally compact measure space (Λ, μ) is determined by a kernel $K(x, y)$ so that the joint intensities of the process can be written as $\det(K(x_i, x_j))$. In particular, if Λ is a finite set and μ is the counting measure on Λ, then K reduces to a $|\Lambda|\times|\Lambda|$-matrix, whose principal minors give the joint densities of the process. The matrix K is not necessarily Hermitian (or real symmetric), even though very often it is.

In the theory of negatively correlated random variables [12, 14], principal minors of a real $n\times n$-matrix give values of a function $\omega : 2^{[n]} \to [0, \infty)$ on the Boolean algebra $2^{[n]}$ of all subsets of $[n] = \{1, 2, \ldots, n\}$. The function ω must be non-negative and must meet the following negative correlation condition. Suppose $y_1, \ldots, y_n$ are indeterminates, and consider the *partition function*

$$Z(\omega; y) \;:=\; \sum_{I\subseteq[n]} \omega(I)y^I, \qquad \text{where} \quad y^I := \prod_{i\in I} y_i.$$

For any positive vector $y = (y_1, \ldots, y_n)$, this determines a probability measure $\mu = \mu_y$ on $2^{[n]}$ by

$$\mu(I) \;:=\; \frac{\omega(I)y^I}{Z(\omega; y)} \qquad \text{for all } \; I \subseteq [n].$$

The *atomic random variables* of this theory are given by $X_i(I) := 1$ if $i \in I$ and 0 otherwise. Their expectations and covariances are

$$\langle X\rangle := \sum_{I\subseteq[n]} X(I)\mu(I) \qquad \text{and} \qquad \mathrm{Cov}(X, Y) := \langle XY\rangle - \langle X\rangle\langle Y\rangle,$$

and the *negative correlation hypothesis* requires that

$$\mathrm{Cov}(X_i, X_j) \leq 0 \qquad \text{for all } \; y > 0, \;\; i \neq j.$$

Wagner [14] asks how to characterize all functions ω satisfying these conditions and arising from some matrix A, i.e., such that $\omega(I) = A_I$ is the minor of A with columns and rows indexed by the subset $I \subseteq [n]$. This application to probability theory is one of the motivations for our algebraic approach to the principal minor assignment problem, namely, the characterization of algebraic relations among principal minors. In this paper we restrict ourselves to the symmetric case.

We now recall a basic fact about Schur complements (e.g. from [1]). The *Schur complement* of an invertible principal submatrix H in a matrix A is the matrix $AH := E - FH^{-1}G$ where

$$A \;=:\; \begin{pmatrix} E & F \\ G & H \end{pmatrix}.$$

The Schur complement is the result of Gaussian elimination applied to reduce the submatrix F to zero using the rows of H. The principal minors of the Schur complement satisfy *Schur's identity*

$$(AH)_\alpha \;=\; \frac{A_{I\cup\alpha}}{A_I} \qquad \text{for all subsets } \alpha \subseteq [n]\backslash I, \tag{5}$$

assuming that H is the principal submatrix of A with rows and columns indexed by I.

The hyperdeterminantal relations of format $2{\times}2{\times}2$ are now derived from Schur's identity (5):

Proof of Theorem 2. The validity of the relation (2) for symmetric $3{\times}3$-matrices is an easy direct calculation. Next suppose that A is a symmetric $n{\times}n$-matrix all of whose principal minors A_I are non-zero. The hyperdeterminantal relation for $I \cup \{j_1, j_2, j_3\}$ coincides with the relation (2) for the principal $3{\times}3$-minor indexed by $\{j_1, j_2, j_3\}$ in the Schur complement AH, after multiplying by A_I^4 to clear denominators. Here we are using Schur's identity (5) for any non-empty subset α of $\{j_1, j_2, j_3\}$. Now, if A is any symmetric matrix that has vanishing principal minors then we write A as the limit of a sequence of matrices whose principal minors are non-zero. The hyperdeterminantal relations hold for every matrix in the sequence, and hence they hold for A as well. □

In Section 5 we offer an alternative proof of Theorem 2, by showing that the vector A_* satisfies the hyperdeterminantal relations of higher-dimensional formats $2{\times}2{\times}\cdots{\times}2$. At this point we note that the hyperdeterminantal relations do not suffice even if all principal minors are non-zero.

Example 4. For $n = 4$ there are 8 hyperdeterminantal relations, one for each facet of the 4-cube:

$$\begin{aligned}
&A_\emptyset^2 A_{123}^2 + A_1^2 A_{23}^2 + A_2^2 A_{13}^2 + A_3^2 A_{12}^2 + 4A_\emptyset A_{12} A_{13} A_{23} + 4A_1 A_2 A_3 A_{123} - 2A_\emptyset A_1 A_{23} A_{123} \\
&- 2A_\emptyset A_2 A_{13} A_{123} - 2A_\emptyset A_3 A_{12} A_{123} - 2A_1 A_2 A_{13} A_{23} - 2A_1 A_3 A_{12} A_{23} - 2A_2 A_3 A_{12} A_{13} = 0, \\
&A_\emptyset^2 A_{124}^2 + A_1^2 A_{24}^2 + A_2^2 A_{14}^2 + A_4^2 A_{12}^2 + 4A_\emptyset A_{12} A_{14} A_{24} + 4A_1 A_2 A_4 A_{124} - 2A_\emptyset A_1 A_{24} A_{124} \\
&- 2A_\emptyset A_2 A_{14} A_{124} - 2A_\emptyset A_4 A_{12} A_{124} - 2A_1 A_2 A_{14} A_{24} - 2A_1 A_4 A_{12} A_{24} - 2A_2 A_4 A_{12} A_{14} = 0, \\
&A_\emptyset^2 A_{134}^2 + A_1^2 A_{34}^2 + A_3^3 A_{14}^2 + A_4^3 A_{13}^2 + 4A_\emptyset A_{13} A_{14} A_{34} + 4A_1 A_3 A_4 A_{134} - 2A_\emptyset A_1 A_{34} A_{134} \\
&- 2A_\emptyset A_3 A_{14} A_{134} - 2A_\emptyset A_4 A_{13} A_{134} - 2A_1 A_3 A_{14} A_{34} - 2A_1 A_4 A_{13} A_{34} - 2A_3 A_4 A_{13} A_{14} = 0, \\
&A_\emptyset^2 A_{234}^2 + A_2^2 A_{34}^2 + A_3^2 A_{24}^2 + A_4^2 A_{23}^2 + 4A_\emptyset A_{23} A_{24} A_{34} + 4A_2 A_3 A_4 A_{234} - 2A_\emptyset A_2 A_{34} A_{234} \\
&- 2A_\emptyset A_3 A_{24} A_{234} - 2A_\emptyset A_4 A_{23} A_{234} - 2A_2 A_3 A_{24} A_{34} - 2A_2 A_4 A_{23} A_{34} - 2A_3 A_4 A_{23} A_{24} = 0,
\end{aligned}$$

$$
\begin{aligned}
A_4^2A_{1234}^2 + A_{14}^2A_{234}^2 + A_{24}^2A_{134}^2 + A_{34}^2A_{124}^2 + 4A_4A_{124}A_{134}A_{234} + 4A_{14}A_{24}A_{34}A_{1234} \\
-2A_4A_{14}A_{234}A_{1234} - 2A_4A_{24}A_{134}A_{1234} - 2A_4A_{34}A_{124}A_{1234} \\
-2A_{14}A_{24}A_{134}A_{234} - 2A_{14}A_{34}A_{124}A_{234} - 2A_{24}A_{34}A_{124}A_{134} \;&=\; 0, \\
A_3^2A_{1234}^2 + A_{13}^2A_{234}^2 + A_{23}^2A_{134}^2 + A_{34}^2A_{123}^2 + 4A_3A_{123}A_{134}A_{234} + 4A_{13}A_{23}A_{34}A_{1234} \\
-2A_3A_{13}A_{234}A_{1234} - 2A_3A_{23}A_{134}A_{1234} - 2A_3A_{34}A_{123}A_{1234} \\
-2A_{13}A_{23}A_{134}A_{234} - 2A_{13}A_{34}A_{123}A_{234} - 2A_{23}A_{34}A_{123}A_{134} \;&=\; 0, \\
A_2^2A_{1234}^2 + A_{12}^2A_{234}^2 + A_{23}^2A_{124}^2 + A_{24}^2A_{123}^2 + 4A_2A_{123}A_{124}A_{234} + 4A_{12}A_{23}A_{24}A_{1234} \\
-2A_2A_{12}A_{234}A_{1234} - 2A_2A_{23}A_{124}A_{1234} - 2A_2A_{24}A_{123}A_{1234} \\
-2A_{12}A_{23}A_{124}A_{234} - 2A_{12}A_{24}A_{123}A_{234} - 2A_{23}A_{24}A_{123}A_{124} \;&=\; 0, \\
A_1^2A_{1234}^2 + A_{12}^3A_{134}^2 + A_{13}^3A_{124}^2 + A_{14}^3A_{123}^3 + 4A_1A_{123}A_{124}A_{134} + 4A_{12}A_{13}A_{14}A_{1234} \\
-2A_1A_{12}A_{134}A_{1234} - 2A_1A_{13}A_{124}A_{1234} - 2A_1A_{14}A_{123}A_{1234} \\
-2A_{12}A_{13}A_{124}A_{134} - 2A_{12}A_{14}A_{123}A_{134} - 2A_{13}A_{14}A_{123}A_{124} \;&=\; 0.
\end{aligned}
$$

The set of solutions to these equations has the correct codimension (five) but it is too big. To illustrate this phenomenon, consider the case when all minors of a given size have the same value:

$$A_\emptyset \;:=\; x_0\,,\;\; A_i \;:=\; x_1\,,\;\; A_{ij} \;:=\; x_2\,,\;\; A_{ijk} \;:=\; x_3\,,\;\; A_{1234} \;:=\; x_4.$$

Under this specialization, the eight hyperdeterminants above reduce to a system of two equations:

$$x_0^2x_3^2 - 6x_0x_1x_2x_3 + 4x_0x_2^3 + 4x_1^3x_3 - 3x_1^2x_2^2 \;=\; x_1^2x_4^2 - 6x_1x_2x_3x_4 + 4x_1x_3^3 + 4x_2^3x_4 - 3x_2^2x_3^2 \;=\; 0. \quad (6)$$

The solution set to these equations in $\mathbb{P}^4$ is the union of two irreducible surfaces, of degree ten and six respectively. The degree ten surface is extraneous and is gotten by requiring additionally that

$$x_0x_3^2 \;=\; x_1^2x_4. \quad (7)$$

The degree six surface is our desired locus of principal minors. It is defined by the two equations

$$3x_2^2 - 4x_1x_3 + x_0x_4 \;=\; 4x_2^3 - 6x_1x_2x_3 + x_0x_3^2 + x_1^2x_4 \;=\; 0. \quad (8)$$

For a concrete numerical example let us consider the symmetric 4×4-matrix

$$A \quad = \quad \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

Its principal minors are $(x_0, x_1, x_2, x_3, x_4) = (1, 2, 3, 4, 5)$ and these satisfy (8) but not (7). On the other hand, the vector $(x_0, x_1, x_2, x_3, x_4) = (1, 2, 3, 4, 4)$ satisfies (6) and (7) but not (8). The corresponding vector $A_* \in \mathbb{R}^{16}$ has all its entries non-zero and satisfies the eight hyperdeterminantal relations but it does not come from the principal minors of any symmetric 4×4-matrix. □

3 A First Converse to Theorem 2

We now derive two additional classes of relations that the principal minors of a symmetric $n \times n$ matrix $A = (a_{ij})$ must satisfy. Throughout this section we set $A_\emptyset = 1$. For $n \geq 4$ and for distinct i, j, k, l, we can write the product $8a_{ij}^2 a_{ik}^2 a_{il}^2 a_{jk} a_{jl} a_{kl}$ in two different ways:

$$\begin{aligned}
&(A_{ijk} - A_{ij}A_k - A_{ik}A_j - A_{jk}A_i + 2A_iA_jA_k)(A_{ijl} - A_{ij}A_l - A_{il}A_j - A_{jl}A_i + 2A_iA_jA_l) \\
&\quad \times (A_{ikl} - A_{ik}A_l - A_{il}A_k - A_{kl}A_i + 2A_iA_kA_l) \\
&= \; 4 \cdot (A_{jkl} - A_{jk}A_l - A_{jl}A_k - A_{kl}A_j + 2A_jA_kA_l)(A_iA_j - A_{ij})(A_iA_k - A_{ik})(A_iA_l - A_{il}).
\end{aligned} \tag{9}$$

Thus conditions (9) are necessary for the realizability of A_*. Note that the $2 \times 2 \times 2$ hyperdeterminantal relations alone imply a weaker version of (9), with both sides squared.

Also for $n \geq 4$ and for distinct i, j, k, l, we define

$$\begin{aligned}
f_{ijkl}(A_*) \;:=\; & A_\emptyset^3 A_{ijkl} - 3 \cdot A_iA_jA_kA_l - A_\emptyset^2 A_iA_{ijk} - A_\emptyset^2 A_jA_{ikl} - A_\emptyset^2 A_kA_{ijl} - A_\emptyset^2 A_lA_{ijk} \\
& + A_\emptyset \cdot (A_iA_jA_{kl} + A_iA_kA_{jl} + A_iA_lA_{jk} + A_jA_kA_{il} + A_jA_lA_{ik} + A_kA_lA_{ij}) \\
& - (A_iA_j - A_\emptyset A_{ij})(A_kA_l - A_\emptyset A_{kl}) - (A_iA_k - A_\emptyset A_{ik})(A_jA_l - A_\emptyset A_{jl}) \\
& - (A_iA_l - A_\emptyset A_{il})(A_jA_k - A_\emptyset A_{jk}).
\end{aligned}$$

This operation extracts the products of entries of A that occur in the determinant A_{ijkl} indexed by permutations of order four. Namely, we find that

$$f_{ijkl}(A_*) \;=\; -2 \cdot (a_{ij}a_{il}a_{jk}a_{kl} + a_{ij}a_{ik}a_{jl}a_{kl} + a_{ik}a_{il}a_{jk}a_{jl}).$$

To rewrite these terms differently, we use the polynomials

$$g_{ijk}(A_*) \;:=\; A_\emptyset^2 A_{ijk} - A_\emptyset \cdot (A_iA_{jk} + A_jA_{ik} + A_kA_{ij}) + 2 \cdot A_iA_jA_k \;=\; 2 \cdot a_{ij}a_{ik}a_{jk},$$

and we observe that

$$\begin{aligned}
g_{ikl}(A_*)g_{ijk}(A_*) &= 4 \cdot a_{ik}^2 a_{ij}a_{il}a_{jk}a_{kl}, \\
g_{ikl}(A_*)g_{ijl}(A_*) &= 4 \cdot a_{il}^2 a_{ij}a_{ik}a_{jl}a_{kl}, \\
g_{ijl}(A_*)g_{ijk}(A_*) &= 4 \cdot a_{ij}^2 a_{ik}a_{il}a_{jk}a_{jl}.
\end{aligned}$$

Using the relation $a_{ij}^2 = A_iA_j - A_{ij}$, we thus obtain

$$\begin{aligned}
& 2f_{ijkl}(A_*)(A_iA_j - A_\emptyset A_{ij})(A_iA_k - A_\emptyset A_{ik})(A_iA_l - A_\emptyset A_{il}) \\
& \quad + g_{ikl}(A_*)g_{ijk}(A_*)(A_iA_j - A_\emptyset A_{ij})(A_iA_l - A_\emptyset A_{il}) \\
& \quad + g_{ikl}(A_*)g_{ijl}(A_*)(A_iA_j - A_\emptyset A_{ij})(A_iA_k - A_\emptyset A_{ik}) \\
& \quad + g_{ijl}(A_*)g_{ijk}(A_*)(A_iA_k - A_\emptyset A_{ik})(A_iA_l - A_\emptyset A_{il}) \;=\; 0.
\end{aligned} \tag{10}$$

By the Schur complement argument from Section 2, we conclude that the relations (10) continue to hold after the substitution

$$\begin{aligned}
& A_\emptyset \mapsto A_I,\; A_i \mapsto A_{I\cup\{i\}},\; A_j \mapsto A_{I\cup\{j\}},\; A_k \mapsto A_{I\cup\{k\}},\; A_l \mapsto A_{I\cup\{l\}}, \\
& A_{ij} \mapsto A_{I\cup\{i,j\}},\; A_{ik} \mapsto A_{I\cup\{i,k\}},\; A_{il} \mapsto A_{I\cup\{i,l\}}, \\
& A_{jk} \mapsto A_{I\cup\{j,k\}},\; A_{jl} \mapsto A_{I\cup\{j,l\}},\; A_{kl} \mapsto A_{I\cup\{k,l\}}, \\
& A_{ijk} \mapsto A_{I\cup\{i,j,k\}},\; A_{ijl} \mapsto A_{I\cup\{i,j,l\}},\; A_{ikl} \mapsto A_{I\cup\{i,k,l\}},\; A_{jkl} \mapsto A_{I\cup\{j,k,l\}}, \\
& A_{ijkl} \mapsto A_{I\cup\{i,j,k,l\}}
\end{aligned} \tag{11}$$

for all subsets $I \subseteq [n] \backslash \{i,j,k,l\}$. We thus proved the following addition to Theorem 2.

Lemma 5. *The principal minors of a symmetric matrix satisfy the conditions (9) and all conditions obtained from (10) via the substitution (11).*

We are finally in a position to prove our first converse to Theorem 2. We assume a nondegeneracy condition to the effect that a subset of the Hadamard-Fischer conditions holds strictly. The condition (12) is equivalent to the condition that *weak sign symmetry* (see, e.g., [8]) holds strictly.

Theorem 6. *Let A_* be a real vector of length 2^n with $A_\emptyset = 1$ that satisfies*

$$A_{I\cap J}A_{I\cup J} < A_I A_J \quad \text{whenever } \#I\cap J = \#I - 1 = \#J - 1. \tag{12}$$

There exists a real symmetric matrix A with principal minors given by A_* if and only if A_* satisfies the hyperdeterminantal relations, relations (9) and all relations obtained from (10) using (11).

Proof. Assuming (12), (9) and (10), we build the real symmetric matrix $A = (a_{ij})$ as follows. The entries A_I indexed by sets I of size 1 and 2 determine all diagonal entries a_{ii} and the magnitudes of all off-diagonal entries. Note that (12) implies that all off-diagonal entries of A are non-zero. It remains to choose the signs of off-diagonal entries correctly. Since the principal minors of A do not change under diagonal similarity $A \mapsto DAD^{-1}$ where each diagonal entry of the matrix D is ± 1, we can fix all first row entries a_{1j} with $j > 1$ to be positive. Then the sign of each entry a_{2j} with $j > 2$ is determined unambiguously by the values A_{12j}, the sign of each entry a_{3j} with $j > 3$ is determined by the values A_{13j}, and so on. In this fashion we prescribe all entries of the matrix A.

Using hyperdeterminantal relations and condition (9), we see that this assignment is consistent with the values of all principal minors A_I where the index set I has size at most 3. Indeed, the hyperdeterminantal relations guarantee that the absolute values of all off-diagonal entries are consistent with the values of all principal minors of order at most 3, whereas conditions (9) guarantee that the signs are consistent as well. The remaining entries of A_*, i.e., the principal minors indexed by sets of size more than 3, are determined uniquely by the already specified entries of A_*. This follows from conditions (10) and from all conditions obtained from (10) via the substitution (11), since in every such condition its largest principal minor occurs linearly with a nonzero coefficient (which is a function of smaller minors) due to inequalities (12). Thus the constructed matrix $A = (a_{ij})$ has the prescribed vector of principal minors A_*. □

4 The Prime Ideal for 4×4-Matrices

From the point of view of algebraic geometry, the following version of our problem is most natural:

Problem 7. *Let P_n be the prime ideal of all homogeneous polynomial relations among the principal minors of a symmetric $n\times n$-matrix. Determine a finite set of generators for the prime ideal P_n.*

The ideal P_n lives in the ring of polynomials in the 2^n unknowns A_I with rational coefficients. For $n = 3$, the ideal I_3 is principal, and its generator is the $2\times2\times2$-hyperdeterminant. In Theorem 2 we identified the subideal H_n which is generated by all hyperdeterminantal relations, one for each 3-dimensional face of the n-cube. For instance, the ideal H_4 for the 4-cube is generated by the eight homogeneous polynomials of degree four in 16 unknowns listed in Example 4. It can be shown that P_n is a minimal prime of the ideal H_n but in general we do not know the minimal generators of P_n.

Two important features of both ideals P_n and H_n is that they are invariant under the symmetry group of the n-cube, and they are homogeneous with respect to the $(n+1)$-dimensional multigrading induced by the n-cube. Both features were used to simplify and organize our computations. In this section we focus on the case $n = 4$, for which we establish the following result.

Theorem 8. *The homogeneous prime ideal P_4 is minimally generated by twenty quartics in the* 16 *unknowns A_I. The corresponding irreducible variety in $\mathbb{P}^{15}$ has codimension five and degree* 96*.*

This theorem was established with the aid of computations using the computer algebra packages `Singular` and `Macaulay 2`. We worked in the polynomial ring with the 5-dimensional multigrading

$$\deg(A_\emptyset) = (1,0,0,0,0),\ \deg(A_1) = (1,1,0,0,0),\ \ldots,\ \deg(A_4) = (1,0,0,0,1)$$
$$\deg(A_{12}) = (1,1,1,0,0), \ldots,\ \deg(A_{234}) = (1,0,1,1,1),\ \deg(A_{1234}) = (1,1,1,1,1).$$

The twenty minimal generators of P_4 come in three symmetry classes, with respect to the symmetry group of the 4-cube (i.e. the Weyl group B_4 of order 384). The three symmetry classes are as follows:

Class 1: The eight 2×2×2 *hyperdeterminants* are listed in Example 4. Their multidegrees are

$$(4,2,2,2,0),\quad (4,2,2,0,2),\quad (4,2,0,2,2),\quad (4,0,2,2,2),$$
$$(4,2,2,2,4),\quad (4,2,2,4,2),\quad (4,2,4,2,2),\quad (4,4,2,2,2).$$

Class 2: There is a unique (up to scaling) minimal generator of P_4 in each of the eight multidegrees

$$(4,2,2,2,1),\quad (4,2,2,1,2),\quad (4,2,1,2,2),\quad (4,1,2,2,2),$$
$$(4,2,2,2,3),\quad (4,2,2,3,2),\quad (4,2,3,2,2),\quad (4,3,2,2,2).$$

The representative for multidegree $(4,3,2,2,2)$ is the following quartic with 40 terms:

$$\begin{gathered}
A_{1234}^2 A_1 A_\emptyset - A_{1234}A_{123}A_{14}A_\emptyset - A_{1234}A_{123}A_1A_4 - A_{1234}A_{124}A_{13}A_\emptyset - A_{1234}A_{124}A_1A_3 \\
-A_{1234}A_{134}A_{12}A_\emptyset - A_{1234}A_{134}A_1A_2 - A_{1234}A_{12}A_{34}A_1 - A_{1234}A_{13}A_{24}A_1 - A_{1234}A_{14}A_{23}A_1 \\
-A_{123}A_{124}A_{13}A_4 - A_{123}A_{124}A_{14}A_3 - A_{123}A_{134}A_{12}A_4 - A_{123}A_{134}A_{14}A_2 - A_{123}A_{234}A_{14}A_1 \\
-A_{123}A_{12}A_{14}A_{34} - A_{123}A_{13}A_{14}A_{24} - A_{124}A_{134}A_{12}A_3 - A_{124}A_{134}A_{13}A_2 - A_{124}A_{234}A_{13}A_1 \\
-A_{124}A_{12}A_{13}A_{34} - A_{124}A_{13}A_{14}A_{23} - A_{134}A_{234}A_{12}A_1 - A_{134}A_{12}A_{13}A_{24} - A_{134}A_{12}A_{14}A_{23} \\
+2A_{1234}A_{12}A_{13}A_4 + 2A_{1234}A_{12}A_{14}A_3 + 2A_{1234}A_{13}A_{14}A_2 + 2A_{123}A_{124}A_{134}A_\emptyset + 2A_{123}A_{124}A_{34}A_1 \\
+2A_{123}A_{134}A_{24}A_1 + 2A_{124}A_{134}A_{23}A_1 + 2A_{234}A_{12}A_{13}A_{14} + A_{1234}A_{234}A_1^2 + A_{123}^2A_{14}A_4 \\
+A_{123}A_{14}^2A_{23} + A_{124}^2A_{13}A_3 + A_{124}A_{13}^2A_{24} + A_{134}^2A_{12}A_2 + A_{134}A_{12}^2A_{34}.
\end{gathered}$$

Class 3: The ideal P_4 contains a four-dimensional space of minimal generators in multidegree $(4,2,2,2,2)$. These generators are not unique but we can choose a symmetric collection of generators by taking the four quartics in the B_4-orbit of the following polynomial with 36 terms:

$$\begin{gathered}
(12)\cdot\big(A_\emptyset^2A_{1234}^2 + A_4^2A_{123}^2 + A_3^2A_{124}^2 + A_{34}^2A_{12}^2 + A_2^2A_{134}^2 + A_{24}^2A_{13}^2 + A_{23}^2A_{14}^2 + A_{234}^2A_1^2\big) \\
-A_\emptyset A_{34}A_{12}A_{1234} - A_\emptyset A_{24}A_{13}A_{1234} - A_\emptyset A_{23}A_{14}A_{1234} - A_\emptyset A_{234}A_1A_{1234} + A_\emptyset A_{234}A_{14}A_{123} \\
+A_\emptyset A_{234}A_{13}A_{124} + A_\emptyset A_{234}A_{134}A_{12} - A_4A_3A_{124}A_{123} - A_4A_2A_{134}A_{123} + A_4A_{23}A_1A_{1234} \\
-A_4A_{23}A_{14}A_{123} + A_4A_{23}A_{13}A_{124} + A_4A_{23}A_{134}A_{12} - A_4A_{234}A_1A_{123} - A_3A_2A_{134}A_{124} \\
+A_3A_{24}A_1A_{1234} + A_3A_{24}A_{14}A_{123} - A_3A_{24}A_{13}A_{124} + A_3A_{24}A_{134}A_{12} - A_3A_{234}A_1A_{124} \\
+A_{34}A_2A_1A_{1234} + A_{34}A_2A_{14}A_{123} + A_{34}A_2A_{13}A_{124} - A_{34}A_2A_{134}A_{12} \\
-A_{34}A_{24}A_{13}A_{12} - A_{34}A_{23}A_{14}A_{12} - A_2A_{234}A_1A_{134} - A_{24}A_{23}A_{14}A_{13}.
\end{gathered}$$

It can be checked, using `Macaulay 2` or `Singular`, that the higher degree polynomials gotten from (9) and (10) by homogenization with $A_\emptyset$ are indeed polynomial linear combinations of these 20 quartics. We thus obtain the following strong converse to Theorem 2 in the case of 4×4-matrices:

Corollary 9. *A vector $A_* \in \mathbb{C}^{16}$ with $A_\emptyset = 1$ can be realized as the principal minors of a symmetric matrix $A \in \mathbb{C}^{4\times 4}$ if and only the above twenty quartics are zero at A_*. If the entries of the given vector A_* are real and satisfy (12) then the entries of the symmetric matrix A are real numbers.*

Proof. Consider the map which takes complex symmetric 4×4-matrices to their vector of principal minors. The image of this map is closed by Corollary 1, and it hence equals the affine variety in $\mathbb{C}^{15} = \{A_\emptyset = 1\}$ defined by the prime ideal P_4. The statement for $\mathbb{R}$ is derived from Theorem 6. □

5 Big Hyperdeterminants and Condensation Polynomials

One ultimate goal is to generalize Theorem 8 and Corollary 9 from $n = 4$ to $n \geq 5$. This section offers first steps in this direction, starting with the general hyperdeterminantal constraints on A_*.

We recall from [4, Chap.14] that the *hyperdeterminant* of a tensor $\mathbf{A} = (\mathbf{a}_{i_1,\ldots,i_r})$ of format $2\times 2\times\cdots\times 2$ is defined as follows. Consider the multilinear form f defined by the tensor $\mathbf{A}$:

$$f(x) \ := \ f(x^{(1)}, x^{(2)} \ldots, x^{(r)}) \ := \ \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_r=0}^{1} \mathbf{a}_{i_1,i_2,\ldots,i_r}\cdot x^{(1)}_{i_1} x^{(2)}_{i_2}\cdots x^{(r)}_{i_r}. \qquad (13)$$

The hyperdeterminant $\det(\mathbf{A})$ is the unique (up to scaling) irreducible polynomial in the entries of $\mathbf{A}$ that characterizes the degeneracy of the form f, i.e., $\det(\mathbf{A}) = 0$ if and only if the equations

$$f(x) \ = \ \frac{\partial f(x)}{\partial x^{(j)}_i} \ = \ 0 \quad \text{for all } \ i,\, j \qquad (14)$$

have a solution $x = (x^{(1)}, x^{(2)}, \ldots, x^{(r)})$ where each $x^{(j)}$ is a non-zero complex vector in $\mathbb{C}^2$.

As before we identify our proposed vector of principal minors $A_* \in \mathbb{R}^{2^n}$ with the corresponding $2\times 2\times\cdots\times 2$-tensor. The following result generalizes Theorem 2 and gives an alternative proof.

Theorem 10. *Let $A = (a_{ij})$ be a symmetric $n\times n$ matrix. Then the tensor A_* of all principal minors of A is a common zero of all the hyperdeterminants of formats from $2\times 2\times 2$ up to $\underbrace{2\times 2\times\cdots\times 2}_{n \text{ terms}}$.*

Proof. It suffices to prove that the highest-dimensional hyperdeterminant vanishes, using Schur complements and induction. Let f be the form (13) corresponding to the tensor A_*. Take

$$x^{(1)} := (a_{12}a_{13} - a_{11}a_{23}\,,\, a_{23}), \quad x^{(2)} := (a_{12}a_{23} - a_{13}a_{22}\,,\, a_{13}), \quad x^{(3)} := (a_{13}a_{23} - a_{12}a_{33}\,,\, a_{12}). \quad (15)$$

and take the remaining $x^{(j)}$ to be $(1, 0)$. With this choice of x, the conditions (14) are satisfied. □

We next introduce the *condensation polynomial* C_n which expresses the determinant $A_{123\cdots n}$ as an algebraic function of the principal minors A_i and A_{ij} of size at most two. For instance, for $n = 3$ the determinant A_{123} is an algebraic function of degree two in $\{A_1, A_2, A_3, A_{12}, A_{13}, A_{23}\}$, and thus C_3 coincides with the 2×2×2-hyperdeterminant. In general, the condensation polynomial C_n is defined as the unique irreducible (and monic in $A_{123\cdots n}$) generator of the principal elimination ideal

$$\langle C_n \rangle \quad = \quad (P_n + \langle A_\emptyset - 1\rangle) \cap \mathbb{Q}[\, A_1, A_2, \ldots, A_n,\, A_{12}, A_{13} \ldots, A_{n-1,n}\,,\, A_{123\cdots n}\,].$$

Using the sign-swapping argument in the proof of Theorem 6, we can show that C_n is a polynomial of degree $2^{\binom{n-1}{2}}$ in $A_{123\cdots n}$. The total degree of C_n is bounded above by $n \cdot 2^{\binom{n-1}{2}}$. The following derivation proves these assertions for $n = 4$, and it illustrates the construction of C_n in general.

Example 11. The condensation polynomial C_4 is an irreducible polynomial of degree 23. It is the sum of 12380 monomials in $\mathbb{Q}[A_1, A_2, A_3, A_4, A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}, A_{1234}]$. To compute C_4, we first consider the following polynomial which expresses the symmetric 4×4-determinant

$$A_{1234} \, - \, \det \begin{pmatrix} A_1 & a_{12} & a_{13} & a_{14} \\ a_{12} & A_2 & a_{23} & a_{24} \\ a_{13} & a_{23} & A_3 & a_{34} \\ a_{14} & a_{24} & a_{34} & A_4 \end{pmatrix} . \tag{16}$$

To get rid of the square roots $a_{ij} = \sqrt{A_i A_j - A_{ij}}$, we swap the sign on the a_{ij} with $1 < i < j$ in all eight possible ways. The orbit of (16) under these sign swaps consists of eight distinct variants of (16). The product of these eight expressions equals C_4. Interestingly, the degree drops to 23. □

We could in fact use the condensation polynomials C_n to replace the hypothesis (12) from the statement of Theorem 6, thus providing a stronger converse to Theorem 2. However, this is not particularly useful for a practical test, since the condensation polynomials C_n are too big to compute explicitly for $n \geq 5$. What is desired instead are explicit generators of the prime ideal P_n.

6 Invariance and the Quartic Generation Conjecture

This section was written after the first version of this paper had been submitted for publication. It is based on discussions with J.M. Landsberg, who suggested Theorem 12 to us in May 2006.

Let $\mathbb{R}[A_\bullet]$ denote the polynomial ring in 2^n unknowns A_I where I runs over all subsets of $\{1, 2, \ldots, n\}$. We are interested in the prime ideal P_n of all homogeneous polynomials in $\mathbb{R}[A_\bullet]$ which constitute algebraic relations among the 2^n principal minors of a generic symmetric $n \times n$-matrix. Clearly, P_n is invariant under the action of the symmetric group S_n on the polynomial ring $\mathbb{R}[A_\bullet]$. In what follows we show that P_n is invariant under a natural Lie group action on $\mathbb{R}[A_\bullet]$ as well. Let $SL_2(\mathbb{R})$ denote group of real 2×2-matrices with determinant 1. The n-fold product of this group,

$$G \quad := \quad SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \times \cdots \times SL_2(\mathbb{R}),$$

acts naturally on the n-fold tensor product

$$\mathbb{R}^{2^n} \quad := \quad \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \cdots \otimes \mathbb{R}^2.$$

Since $\mathbb{R}[A_\bullet]$ is the ring of polynomial functions of $\mathbb{R}^{2^n}$, we get an action of G on $\mathbb{R}[A_\bullet]$.

Theorem 12. *The homogeneous prime ideal P_n is invariant under the group G.*

Proof. Consider an $n \times 2n$-matrix $(B \; C)$ where B and C are generic $n \times n$-matrices subject to the constraint that $B^{-1}C$ is symmetric. We identify each principal minor of the symmetric matrix $B^{-1}C$ with an $n \times n$-minor of the matrix $(B \; C)$. The 2^n maximal minors of $(B \; C)$ which come from principal minors of $B^{-1}C$ are precisely those whose column index sets $J \subset \{1, 2, \ldots, 2n\}$ satisfy $\#(J \cap \{i, i+n\}) = 1$ for $i = 1, 2, \ldots, n$. We call these the *special maximal minors* of $(B \; C)$. The prime ideal P_n consists of the algebraic relations among the 2^n special maximal minors.

We need to show that the parametric variety corresponding to the prime ideal P_n is invariant under the group G. In order to do this, we consider the representation of the group G by $2n \times 2n$-matrices of the form

$$g \;=\; \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

where D_1, D_2, D_3, D_4 are diagonal $n \times n$-matrices which satisfy the identity

$$D_1 \cdot D_4 \,-\, D_2 \cdot D_3 \quad = \quad \mathbf{1} \qquad\qquad \text{(the } n\times n\text{-identity matrix)}$$

Here, the i-th factor $SL_2(\mathbb{R})$ in the n-fold product G corresponds to the 2×2-matrix formed by the entries in position (i,i) of the diagonal matrices D_1, D_2, D_3, D_4. Let g be any element of G represented by a $2n\times 2n$-matrix. Then the vector in $\mathbb{R}^{2^n}$ of special maximal minors of $(B\ C) \cdot g$ is precisely the result of applying g to the vector in $\mathbb{R}^{2^n}$ of special maximal minors of $(B\ C)$.

It now suffices to show is the following matrix-theoretic statement: if $B, C, \tilde{B}, \tilde{C}$ are $n \times n$-matrices such that $(\tilde{B}\ \tilde{C}) = (B\ C) \cdot g$ for some $g \in G$, and if $B^{-1}C$ is symmetric, then $\tilde{B}^{-1}\tilde{C}$ is also symmetric. The following lemma proves this statement. □

Lemma 13. *Let B and C be invertible $n \times n$-matrix such that $B^{-1}C$ is symmetric, and let D_1, D_2, D_3, D_4 be diagonal $n \times n$-matrices satisfying $D_1 \cdot D_4 \,-\, D_2 \cdot D_3 \,=\, \mathbf{1}$. Then the matrix $(BD_1 + CD_2)^{-1}(BD_3 + CD_4)$ is also symmetric.*

Proof. The assumption that $B^{-1}C$ is symmetric is equivalent to the following identity:

$$B \cdot C^T \,-\, C \cdot B^T \quad = \quad \mathbf{0}. \tag{17}$$

Similarly, the desired conclusion states that the following difference is the zero matrix $\mathbf{0}$:

$$(BD_1 + CD_2) \cdot (BD_3 + CD_4)^T \,-\, (BD_3 + CD_4) \cdot (BD_1 + CD_2)^T$$

Multiplying out, cancelling common terms, and using both $D_i^T = D_i$ and the identity (17), we simplify the above matrix expression as follows:

$$\begin{aligned} & CD_2D_3B^T + BD_1D_4C^T - BD_3D_2C^T - CD_4D_1B^T \\ = \;& B \cdot (D_1D_4 - D_2D_3) \cdot C^T \,-\, C \cdot (D_1D_4 - D_2D_3) \cdot B^T \\ = \;& \quad B\mathbf{1}C^T - C\mathbf{1}B^T \quad = \quad BC^T - CB^T \quad = \quad \mathbf{0}. \end{aligned}$$

This proves the lemma and hence the theorem. □

We define the *hyperdeterminantal module* to the G-orbit of the $2\times2\times2$-hyperdeterminant under the action of the group G and the symmetric group S_n. This orbit is a subspace of the finite-dimensional vector space $\mathbb{R}[A_\bullet]_4$ of quartic polynomials in the 2^n unknowns A_I. Using a representation theoretic argument, it can be shown that the vector space dimension of the hyperdeterminantal module equals

$$\binom{2^{n-3}+3}{4} \cdot \binom{n}{3}$$

This number is one for $n = 3$, and it is 20 for $n = 4$. We propose the following natural conjecture.

Conjecture 14. *The prime ideal P_n is generated by the hyperdeterminantal module.*

For $n = 3$, the prime ideal is principal and generated by the $2\times2\times2$-hyperdeterminant. For $n = 4$, this conjecture is established by our computations in Section 4.

Acknowledgements

We thank David Wagner and Yuval Peres for information on the probabilistic aspects of the principal minor assignment problem. Theorem 12 was suggested to us by J.M. Landsberg, in response to the first version of this paper. We thank him for his permission to include this result here. Bernd Sturmfels was supported by the National Science Foundation (DMS-0456960).

References

[1] Richard A. Brualdi and Hans Schneider. Determinantal identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir, and Cayley. *Linear Algebra Appl.*, 5253:769–791, 1983.

[2] Gernot M. Engel and Hans Schneider. The Hadamard-Fischer inequality for a class of matrices defined by eigenvalue monotonicity. *Linear and Multilinear Algebra*, 4(3):155–176, 197677.

[3] Shmuel Friedland. On inverse multiplicative eigenvalue problems for matrices. *Linear Algebra and Appl.*, 12(2):127–137, 1975.

[4] Israel M. Gel′fand, Mikhail M. Kapranov, and Andrei V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants.* Mathematics: Theory & Applications. Birkhäuser, Boston MA, 1994.

[5] Gianluca Gorni, Halszka Tutaj-Gasińska, and Gaetano Zampieri. Drużkowski matrix search and D-nilpotent automorphisms. *Indag. Math. (N.S.)*, 10(2):235–245, 1999.

[6] Kent Griffin and Michael J. Tsatsomeros. Principal minors, part I: a method for computing all the principal minors of a matrix. *Linear Algebra Appl.*, 419(1):107–124, 2006.

[7] Kent Griffin and Michael J. Tsatsomeros. Principal minors, part II: the principal minor assignment problem. *Linear Algebra Appl.*, 419(1):125–171, 2006.

[8] Olga Holtz. Not all GKK τ-matrices are stable. *Linear Algebra Appl.*, 291(1-3):235–244, 1999.

[9] Olga Holtz and Hans Schneider. Open problems on GKK τ-matrices. *Linear Algebra Appl.*, 345:263–267, 2002.

[10] Ben J. Hough, Manjunath Krishnapur, Yuval Peres and Bálint Virág. Determinantal processes and independence. *Probab. Surv.*, 3:206–229, 2006.

[11] Odile Macchi. The coincidence approach to stochastic point processes. *Advances in Appl. Probability*, 7:83–122, 1975.

[12] Robin Pemantle. Towards a theory of negative dependence. *J. Math. Phys.*, 41(3):1371–1390, 2000.

[13] Michael J. Tsatsomeros and Lei Li. A recursive test for P-matrices. *BIT*, 40(2):410–414, 2000.

[14] David G. Wagner. Negatively correlated random variables and Mason's conjecture. ArXiv:math.CO0602648, 2006.