

The inverse eigenvalue problem for symmetric anti-bidiagonal matrices

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Abstract

The inverse eigenvalue problem for real symmetric matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & * \\ 0 & 0 & 0 & \cdots & 0 & * & * \\ 0 & 0 & 0 & \cdots & * & * & 0 \\ & & & & & & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ & & & & & & \\ 0 & 0 & * & \cdots & 0 & 0 & 0 \\ 0 & * & * & \cdots & 0 & 0 & 0 \\ * & * & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

is solved. The solution is shown to be unique. The problem is also shown to be equivalent to the inverse eigenvalue problem for a certain subclass of Jacobi matrices.

1 Introduction

The goal of this paper is to characterize completely the spectra of real symmetric *anti-bidiagonal* matrices, i.e., matrices of the form

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_n \\ 0 & 0 & \cdots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n-2} & \cdots & 0 & 0 \\ a_n & a_{n-1} & \cdots & 0 & 0 \end{bmatrix}, \quad a_1, \dots, a_n \in \mathbb{R}. \quad (1)$$

This work is motivated by the author's ongoing work on the nonnegative inverse eigenvalue problem and is inspired by well-known results on Jacobi matrices due to Hochstadt [5], [6], Hald [4], Gray and Wilson [3], as well as by the classical connection between the Jacobi matrices and orthogonal polynomials (see, e.g., [1, p. 267]).

The blanket assumption for the rest of the paper is that all a_j are positive. This restriction is clearly unimportant, since the sign of any a_j , $j > 1$, can be changed using a unitary similarity of the form

$$\text{diag}(\varepsilon_1, \dots, \varepsilon_n), \quad \varepsilon_j = \pm 1,$$

and the problem for $a_1 < 0$ can be solved by switching from A to $-A$. The assumption $a_j > 0$, $j = 1, \dots, n$, is however just right to guarantee uniqueness of a matrix that realizes a given n -tuple as its spectrum.

2 Definitions and notation

Notation used in the paper is rather standard. The spectrum of a matrix A is denoted by $\sigma(A)$. A submatrix of A with rows indexed by an increasing sequence α and columns indexed by another sequence β is denoted by $A(\alpha, \beta)$. For simplicity, a principal submatrix of A with rows and columns indexed by α is denoted by $A(\alpha)$. (A typical choice for such an α will be $i:j$, the sequence of consecutive integers i through j .) The size of a sequence α is denoted by $\#\alpha$. If $\#\alpha = \#\beta$, then $\det A(\alpha, \beta)$ is denoted by $A[\alpha, \beta]$; $\det A(\alpha)$ is denoted by $A[\alpha]$. The elementary symmetric functions of an n -tuple $\Lambda =: (\lambda_1, \dots, \lambda_n)$ are denoted as $\sigma_j(\Lambda)$. Thus

$$\sigma_1(\Lambda) := \sum_j \lambda_j, \quad \sigma_2(\Lambda) := \sum_{i < j} \lambda_i \lambda_j, \quad \text{etc.}$$

The term anti-bidiagonal matrix was already introduced. Other requisite definitions are listed next.

A *Jacobi matrix* is a tridiagonal matrix with positive codiagonal entries.

A *sign-regular matrix of class $d \leq n$ with signature sequence $\varepsilon_1, \dots, \varepsilon_d$* , where $\varepsilon_j = \pm 1$ for all j , is a matrix satisfying

$$\varepsilon_j A[\alpha] \geq 0 \quad \text{whenever } \#\alpha = j, \quad j = 1, \dots, d.$$

If in addition all minors of order at most d are nonzero, the matrix is called *strictly sign-regular*. Finally, if a certain power of a sign-regular matrix of class d is strictly sign-regular, then the matrix is called sign-regular of class d^+ . A particular case of strict sign regularity is *total positivity* when all minors of a matrix are positive.

A sequence $\mu_1 < \dots < \mu_k$ is said to *interlace* a sequence $\lambda_1 < \dots < \lambda_{k+1}$ if

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \dots < \mu_k < \lambda_{k+1}.$$

3 Results

The following theorem is the main result of this paper.

Theorem 1 *A real n -tuple Λ can be realized as the spectrum of a symmetric anti-bidiagonal matrix (1) with all a_j positive if and only if $\Lambda = (\lambda_1, \dots, \lambda_n)$ where*

$$\lambda_1 > -\lambda_2 > \lambda_3 > \dots > (-1)^{n-1} \lambda_n > 0. \quad (2)$$

The realizing matrix is necessarily unique.

Proof. Necessity. Let J denote the antidiagonal unit matrix

$$J := \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \cdot & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Note that J is sign-regular of class n with the signature sequence

$$1, -1, -1, 1, 1, \dots, (-1)^{\lceil n-1/2 \rceil}. \quad (3)$$

Next, note that $B := JA$ is a nonnegative bidiagonal matrix, hence all its minors are nonnegative. Now, by the Cauchy-Binet formula

$$A[\alpha] = (JB)[\alpha] = \sum_{\#\beta=\#\alpha} J[\alpha, \beta]B[\beta, \alpha].$$

We conclude that the matrix A is sign-regular of class n with the same signature sequence (3). Since A^2 is a positive definite Jacobi matrix, a high enough power of A^2 is totally positive, hence A is sign-regular of type n^+ .

By a theorem of Gantmacher and Krein [2, p. 301], the eigenvalues of A therefore can be arranged to form a sequence with alternating signs and strictly decreasing absolute values whose first element is positive, i.e., the spectrum $\sigma(A)$ satisfies (2).

Sufficiency. First reduce the inverse problem for anti-bidiagonal matrices to the inverse problem for certain Jacobi matrices. Consider a matrix of the form (1). To stress its dependence on n parameters a_1 through a_n , let us denote it by A_n . The argument will involve the collection of all matrices A_n , $n \in \mathbb{Z}$, determined by a single sequence a_1, a_2, \dots . Denote the characteristic polynomial of A_n by p_n :

$$p_n(\lambda) := \det(\lambda I - A_n).$$

Expanding it by its first row yields

$$p_n(\lambda) = \lambda p_{n-1}(\lambda) - a_n^2 p_{n-2}(\lambda), \quad n \geq 2 \quad (4)$$

$$p_0(\lambda) = 1, \quad p_1(\lambda) = \lambda - a_1, \quad (5)$$

since the matrix A_{n-1} is similar to its reflection about the antidiagonal.

This three-term recurrence relation (4) with initial conditions (5) is also satisfied (see, e.g., [1, p. 267] or check directly) by the characteristic polynomials of the Jacobi matrices

$$B_n := \begin{bmatrix} a_1 & a_2 & 0 & \cdots & 0 & 0 \\ a_2 & 0 & a_3 & \cdots & 0 & 0 \\ 0 & a_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_n \\ 0 & 0 & 0 & \cdots & a_n & 0 \end{bmatrix} \quad (6)$$

if each of them is expanded by its last row. Thus the inverse eigenvalue problem for anti-bidiagonal matrices A_n is equivalent to the inverse eigenvalue problem for Jacobi matrices B_n .

Now comes the crucial step in the proof. Consider expanding the characteristic polynomials of matrices B_n in the opposite order, i.e., starting from the first row. Precisely, let us denote by q_{n-j+1} the characteristic polynomial of the principal submatrix $B_n(j:n)$, with $q_n = p_n$. The corresponding recurrence relation is

$$q_n(\lambda) = (\lambda - a_1)q_{n-1}(\lambda) - a_2^2 q_{n-2}(\lambda), \quad (7)$$

$$q_{n-j}(\lambda) = \lambda q_{n-j-1}(\lambda) - a_{j+2}^2 q_{n-j-2}(\lambda), \quad j = 1, \dots, n-2, \quad (8)$$

$$q_0(\lambda) = 1, \quad q_1(\lambda) = \lambda. \quad (9)$$

Let Λ be an n -tuple satisfying (2). Define the polynomial q_n as

$$q_n(\lambda) := \prod_{j=1}^n (\lambda - \lambda_j)$$

and show that one can define polynomials q_{n-j} for all $j = 1, \dots, n$ so as to meet the requirements (7)–(9). To this end, first define

$$a_1 := \sigma_1(\Lambda), \quad q_{n-1}(\lambda) := \frac{(-1)^n q_n(-\lambda) - q_n(\lambda)}{2a_1}. \quad (10)$$

Note that $a_1 > 0$ due to the properties of Λ and that the (monic) polynomial q_{n-1} is even or odd depending on whether $n-1$ is even or odd. Also note that the coefficient of λ^{n-3} in q_{n-1} is equal to

$$\frac{\sigma_3(\Lambda)}{a_1} = \frac{\sigma_3(\Lambda)}{\sigma_1(\Lambda)} < 0.$$

On the other hand, the coefficient of λ^{n-2} in $q_n(\lambda)$ is $\sigma_2(\Lambda) < 0$. Therefore, it remains to show that the quantity $\frac{\sigma_3(\Lambda)}{\sigma_1(\Lambda)} - \sigma_2(\Lambda)$ is positive, so a_2 can be defined as its (positive) square root:

$$a_2 := \sqrt{\frac{\sigma_3(\Lambda)}{\sigma_1(\Lambda)} - \sigma_2(\Lambda)}.$$

Indeed, let us prove that

$$\sigma_3(\Lambda) > \sigma_1(\Lambda)\sigma_2(\Lambda) \quad (11)$$

by induction. The base case is $n = 3$, where $\lambda_1 > -\lambda_2 > \lambda_3 > 0$. Then (11) reduces to the inequality

$$\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) (\lambda_1 + \lambda_2 + \lambda_3) > 1. \quad (12)$$

Differentiating the left-hand side of (12), one can check that it is an increasing function of λ_1 for $\lambda_1 \geq -\lambda_2$. Since the left-hand side is exactly 1 when $\lambda_1 = -\lambda_2$, this proves (12) and therefore proves (11). If $n > 3$, also notice that inequality (11) turns into equality for $\lambda_1 = -\lambda_2$, so it remains to argue that the difference $\sigma_3(\Lambda) - \sigma_1(\Lambda)\sigma_2(\Lambda)$ is an increasing function of λ_1 for $\lambda_1 \geq -\lambda_2$. But this is indeed the case, as can be seen by considering symmetric functions of the set $\Lambda' := (-\lambda_2, \dots, -\lambda_n)$. Since

$$\sigma_1(\Lambda) = \lambda_1 - \sigma_1(\Lambda'), \quad \sigma_2(\Lambda) = -\lambda_1\sigma_1(\Lambda') + \sigma_2(\Lambda'), \quad \sigma_3(\Lambda) = \lambda_1\sigma_2(\Lambda') - \sigma_3(\Lambda'),$$

the inequality (11) amounts to

$$\lambda_1^2 \sigma_1(\Lambda') - \lambda_1 \sigma_1^2(\Lambda') + \sigma_1(\Lambda') \sigma_2(\Lambda') - \sigma_3(\Lambda') > 0,$$

and the derivative of the last left-hand side is positive, since $\lambda_1 \geq \sigma_1(\Lambda')$. This completes the proof of (11). Thus, a_2 is well-defined.

With these definitions in place, define q_{n-2} from (7), i.e., let

$$q_{n-2}(\lambda) := -\frac{q_n(\lambda) - (\lambda - a_1)q_{n-1}(\lambda)}{a_2^2}.$$

Note that q_{n-2} is a monic polynomial and is odd or even (precisely, it has the same parity as its leading term).

Now show that the roots of q_{n-1} interlace those of q_n and the roots of q_{n-2} interlace those of q_{n-1} . Note that the polynomials $q_n(\lambda)$ and $(-1)^{n-1}q_n(-\lambda)$ have the same sign on the intervals

$$(-|\lambda_1|, -|\lambda_2|), (-|\lambda_3|, -|\lambda_4|), \dots, (|\lambda_2|, |\lambda_1|).$$

Moreover, the sequence of these signs is alternating. The polynomial q_{n-1} defined by (10) therefore has exactly $n - 1$ real zeros, each of them between two consecutive zeros of q_n . The implication for the root interlacing of q_{n-2} and q_{n-1} is immediate and is a standard argument on orthogonal polynomials (cf. [1, Section 5.4]): Due to the root interlacing of q_{n-1} and q_n and due to (7), the values of q_{n-2} at the zeros of q_{n-1} form an alternating sequence. Therefore, the roots of q_{n-2} interlace those of q_{n-1} .

The rest of the argument is quite straightforward. With q_{n-j} and q_{n-j-1} defined, one defines q_{n-j-2} from (8) making sure that a_{j+2}^2 is indeed positive, for each $j = 1, \dots, n - 2$. The resulting monic polynomials will have alternating parities and interlacing roots. The quantity a_{j+2}^2 is to be set equal to the difference between the second elementary symmetric function σ_2 of the roots of q_{n-j-1} and the second elementary symmetric function of the roots of q_{n-j} . With a slight abuse of notation, this may be denoted by

$$a_{j+2}^2 = \sigma_2(q_{n-j-1}) - \sigma_2(q_{n-j}).$$

The roots of either polynomial are symmetric about 0, therefore, the corresponding second elementary symmetric function is simply

$$(-1) \cdot \text{the sum of squares of all positive roots.}$$

By the interlacing property, the sum of squares for q_{n-j} exceeds that for q_{n-j-1} , hence $\sigma_2(q_{n-j-1}) - \sigma_2(q_{n-j}) > 0$ and hence a_{j+2} is well defined.

The argument also shows the uniqueness of the realizing matrix (6), therefore the uniqueness of the realizing matrix (1), provided, of course, that a_j are chosen to be positive. \square

The following corollary was established in the course of the above proof.

Corollary 2 *A real n -tuple Λ can be realized as the spectrum of a Jacobi matrix (6) if and only if $\Lambda = (\lambda_1, \dots, \lambda_n)$ where*

$$\lambda_1 > -\lambda_2 > \lambda_3 > \dots > (-1)^{n-1} \lambda_n > 0.$$

The realizing matrix is necessarily unique.

Finally, another simple consequence of Theorem 1 is the following result.

Corollary 3 *Let \mathcal{M} be a real positive n -tuple. Then there exists a Jacobi matrix that realizes \mathcal{M} as its spectrum and has a symmetric anti-bidiagonal square root of the form (1) with all a_j positive.*

Proof. Let the elements of \mathcal{M} be ordered $\mu_1 > \mu_2 > \cdots > \mu_n (> 0)$. Define

$$\lambda_j := (-1)^{j-1} \sqrt{\mu_j}, \quad j = 1, \dots, n, \quad \Lambda := (\lambda_j : j = 1, \dots, n).$$

Then

$$\lambda_1 > -\lambda_2 > \lambda_3 > \cdots > (-1)^{n-1} \lambda_n > 0.$$

By Theorem 1, there exists a symmetric anti-bidiagonal matrix A with spectrum $\sigma(A) = \Lambda$. But then $B := A^2$ is a Jacobi matrix with spectrum \mathcal{M} . \square

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