ON CONVERGENCE OF INFINITE MATRIX PRODUCTS

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Abstract. A necessary and sufficient condition for the convergence of an infinite right product of matrices of the form

$$A := \begin{bmatrix} I & B \\ 0 & C \end{bmatrix},$$

with (uniformly) contracting submatrices C, is proven.

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1. Introduction. Consider the set of all matrices in $\mathbb{C}^{d\times d}$ of the form

$$A := \begin{bmatrix} I_s & B \\ 0 & C \end{bmatrix}, \tag{1.1}$$

where I_s denotes the identity matrix of order $s < d$.

Matrices (1.1) are known (e.g., [1]) to form an LCP set whenever the submatrices B are uniformly bounded and the submatrices C are uniformly contracting, that is, satisfy the condition $\|C\| \le r$ for some fixed matrix (i.e., submultiplicative) norm $\|\cdot\|$ on $\mathbb{C}^{(d-s)\times(d-s)}$ and some constant $r < 1$. To recall, a set Σ has the LCP (RCP) property if all left (right) infinite products formed from matrices in Σ are convergent.

Matrices of the form (1.1), with uniformly bounded submatrices B and uniformly contracting submatrices C, do not necessarily form an RCP set. (They do form such a set if and only if they satisfy a very stringent condition given in Corollary 2.3 below.) However, there exists a simple criterion that can be used to check whether a *particular* right infinite product formed from such matrices converges.

2. A convergence test. THEOREM 2.1. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of matrices of the form (1.1) and let*

$$\|C_n\| \le r < 1 \qquad \text{for all} \quad n \in \mathbb{N}$$

for some matrix norm $\|\cdot\|$. The sequence $(P_n := A_1A_2\cdots A_n)$ converges if and only if so does the sequence $(B_n(I - C_n)^{-1})$. In this event,

$$\lim_{n\to\infty} P_n = \begin{bmatrix} I & \lim_{n\to\infty} B_n(I - C_n)^{-1} \\ 0 & 0 \end{bmatrix}.$$

Proof. To prove the necessity, partition P_n conformably with A_n. Then

$$P_n = \begin{bmatrix} I & X_n \\ 0 & C_1C_2\cdots C_n \end{bmatrix} \qquad \text{where} \qquad X_n := \sum_{i=0}^{n} B_{n-i}(C_{n+1-i}C_{n+2-i}\cdots C_n).$$

If (P_n) converges, then $\lim_{n\to\infty}(X_n - X_{n-1}) = 0$. Also, $\|(I - C_n)^{-1}\| \le 1/(1 - r)$ for all $n \in \mathbb{N}$. But $X_n = B_n + X_{n-1}C_n$, so

$$B_n(I - C_n)^{-1} - X_{n-1} = (X_n - X_{n-1})(I - C_n)^{-1} \xrightarrow[n\to\infty]{} 0.$$

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Hence

$$\lim_{n\to\infty} B_n(I-C_n)^{-1} = \lim_{n\to\infty} X_n.$$

Now prove the sufficiency. Without loss of generality one can assume that $s = d-s$. Indeed, simply replace each A_n by

$$\widetilde{A_n} := \begin{bmatrix} I_{\max\{s,d-s\}} & \widetilde{B_n} \\ 0 & \widetilde{C_n} \end{bmatrix}$$

where

$$\widetilde{B_n} := \begin{cases} \begin{bmatrix} B_n & 0_{s\times(2s-d)} \end{bmatrix} & \text{if } s \geq d-s \\ \begin{bmatrix} B_n \\ 0_{(d-2s)\times(d-s)} \end{bmatrix} & \text{if } s < d-s \end{cases},$$

$$\widetilde{C_n} := \begin{cases} \begin{bmatrix} C_n & 0_{(d-s)\times(2s-d)} \\ 0_{(2s-d)\times(d-s)} & 0_{2s-d} \end{bmatrix} & \text{if } s \geq d-s \\ C_n & \text{if } s < d-s \end{cases}.$$

Then the matrices $\widetilde{A_n}$ satisfy all the assumptions of the theorem and the sequence $(B_n(I-C_n)^{-1})$ (the product P_n) converges iff so does the sequence $(\widetilde{B_n}(I-\widetilde{C_n})^{-1})$ (the product $\widetilde{P_n}$).

Thus, assume that $s = d-s$. Note that if the sequence $(B_n(I-C_n)^{-1})$ converges, then the sequence (B_n) is bounded, since $\|I-C_n\| \leq 1+r$ for all n. Now, let

$$\begin{aligned} D_n &:= X_n - B_n(I-C_n)^{-1} \\ Y_n &:= B_{n+1}(I-C_{n+1})^{-1} - B_n(I-C_n)^{-1} \end{aligned}$$

for all $n \in \mathbb{N}$. Then

$$D_{n+1} = (D_n - Y_n)C_{n+1}, \tag{2.1}$$

hence

$$\|D_{n+1}\| \leq (\|D_n\| + \|Y_n\|)\|C_{n+1}\| \leq (\|D_n\| + \|Y_n\|)r.$$

Repeated use of this inequality gives

$$\|D_n\| \leq \sum_{i=1}^{n-1} \|Y_{n-i}\| r^i.$$

This implies, in particular, that

$$S := \limsup_{n\to\infty} \|D_n\| < \infty.$$

Since $\lim_{n\to\infty} Y_n = 0$, the identity (2.1) and the upper bound on $\|C_n\|$ imply that $S \leq rS$, therefore $S = 0$, that is,

$$\lim_{n\to\infty} D_n = 0.$$

□

The obtained criterion of convergence can be used to make two more observations in the same spirit.

COROLLARY 2.2. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of matrices of the form (1.1) such that the sequence (C_n) converges to a matrix C with spectral radius smaller than 1. Then the sequence $(P_n := A_1A_2\cdots A_n)$ converges if and only if so does the sequence (B_n). In this event,*

$$\lim_{n\to\infty} P_n = \begin{bmatrix} I & \lim_{n\to\infty} B_n(I-C)^{-1} \\ 0 & 0 \end{bmatrix}.$$

Proof. If $\varrho(C) < 1$, then there exists a matrix norm $\|\cdot\|$ on $\mathbb{C}^{(d-s)\times(d-s)}$ such that $\|C\| < 1$ (e.g., [p.297, Lemma 5.6.10][2]). So, $\|C_n\| \le r$ for all $n \ge N$ for some $r < 1$ and some $N \in \mathbb{N}$, so the assumption of the theorem is then satisfied. The product P_n converges whenever so does the product $A_N A_{N+1}\cdots$, so (P_n) has a limit whenever (B_n) has one. By the same reason, the sequence $((I-C_n)^{-1})_{n=N}^{\infty}$ is bounded, so the necessity argument from the proof of the Theorem shows that the convergence of (B_n) is also necessary. ∎

COROLLARY 2.3. *A set Σ consisting of matrices of the form (1.1,) with uniformly contracting submatrices C, is an RCP set if and only if*

$$B_1(I-C_1)^{-1} = B_2(I-C_2)^{-1} \qquad \text{for all} \quad A_1,\ A_2 \in \Sigma, \tag{2.2}$$

where

$$A_i = \begin{bmatrix} I & B_i \\ 0 & C_i \end{bmatrix}, \qquad i = 1, 2.$$

Proof. Given $A_1,\ A_2 \in \Sigma$, apply Theorem 2.1 to the product $A_1A_2A_1A_2\cdots$ to see that the condition (2.2) is necessary and sufficient for the convergence of such a product. But if it is satisfied for all pairs of matrices from Σ, then it is sufficient for the convergence of any right product of matrices from Σ. ∎

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