Nonlinear Optimization for Optimal Control

Pieter Abbeel
UC Berkeley EECS

Many slides and figures adapted from Stephen Boyd

[optional] Boyd and Vandenberghe, Convex Optimization, Chapters 9 – 11
[optional] Betts, Practical Methods for Optimal Control Using Nonlinear Programming

Bellman's curse of dimensionality

- n-dimensional state space
- Number of states grows exponentially in n (assuming some fixed number of discretization levels per coordinate)
- In practice
  - Discretization is considered only computationally feasible up to 5 or 6 dimensional state spaces even when using
    - Variable resolution discretization
    - Highly optimized implementations

This Lecture: Nonlinear Optimization for Optimal Control

- Goal: find a sequence of control inputs (and corresponding sequence of states) that solves:

$$\begin{aligned} \min_{u,x} \quad & \sum_{t=0}^{H} g(x_t, u_t) \\ \text{subject to} \quad & x_{t+1} = f(x_t, u_t) \quad \forall t \\ & u_t \in \mathcal{U}_t \quad \forall t \\ & x_t \in \mathcal{X}_t \quad \forall t \end{aligned}$$

- Generally hard to do. We will cover methods that allow to find a local minimum of this optimization problem.
- Note: iteratively applying LQR is one way to solve this problem if there were no constraints on the control inputs and state.
- In principle (though not in our examples), u could be parameters of a control policy rather than the raw control inputs.

Outline

- **Unconstrained minimization**
  - **Gradient Descent**
  - Newton's Method
- Equality constrained minimization
- Inequality and equality constrained minimization

Unconstrained Minimization

$$\min_x f(x) \qquad (1)$$

(Implicitly assumed x can be chosen from the entire domain of f, often $\mathbb{R}^n$.)

- If x* satisfies:

$$\begin{aligned} \nabla_x f(x^*) &= 0 \quad (2) \\ \nabla_x^2 f(x^*) &\succeq 0 \quad (3) \end{aligned}$$

then x* is a local minimum of f.

- In simple cases we can directly solve the system of n equations given by (2) to find candidate local minima, and then verify (3) for these candidates.
- In general however, solving (2) is a difficult problem. Going forward we will consider this more general setting and cover numerical solution methods for (1).

Steepest Descent

- Idea:
  - Start somewhere
  - Repeat: Take a small step in the steepest descent direction

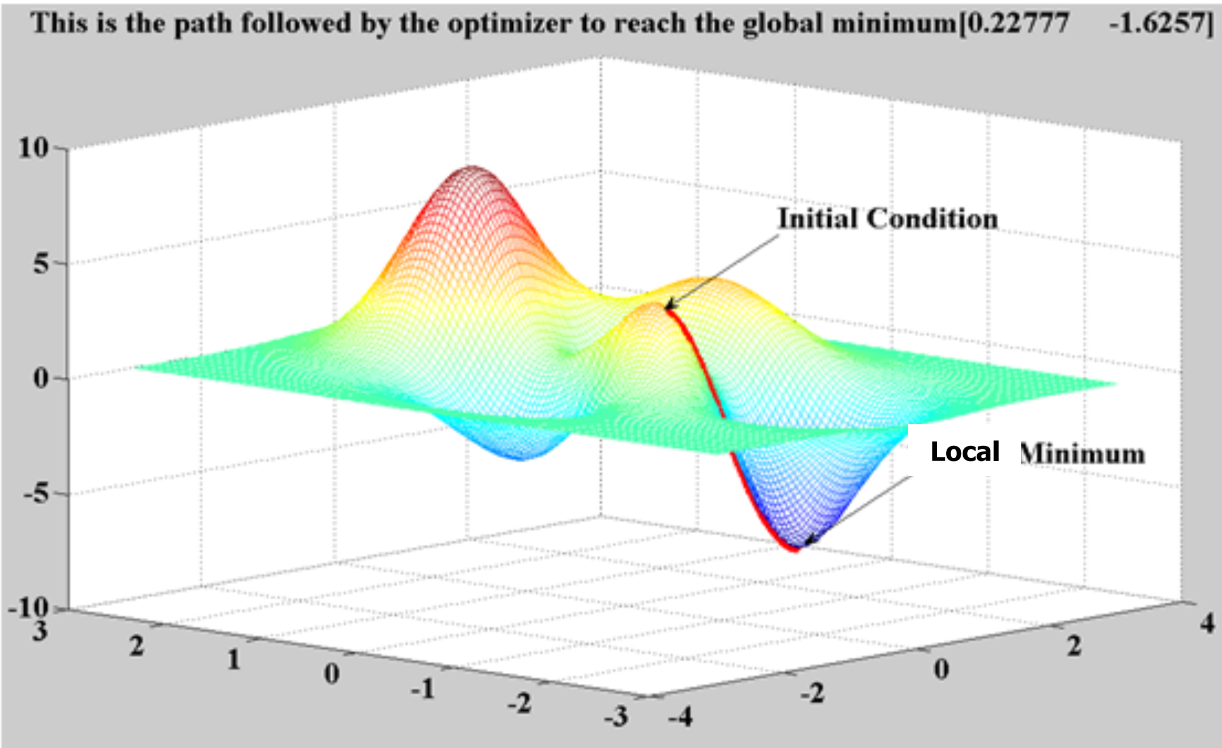


Figure source: Mathworks

Steep Descent

- Another example, visualized with contours:

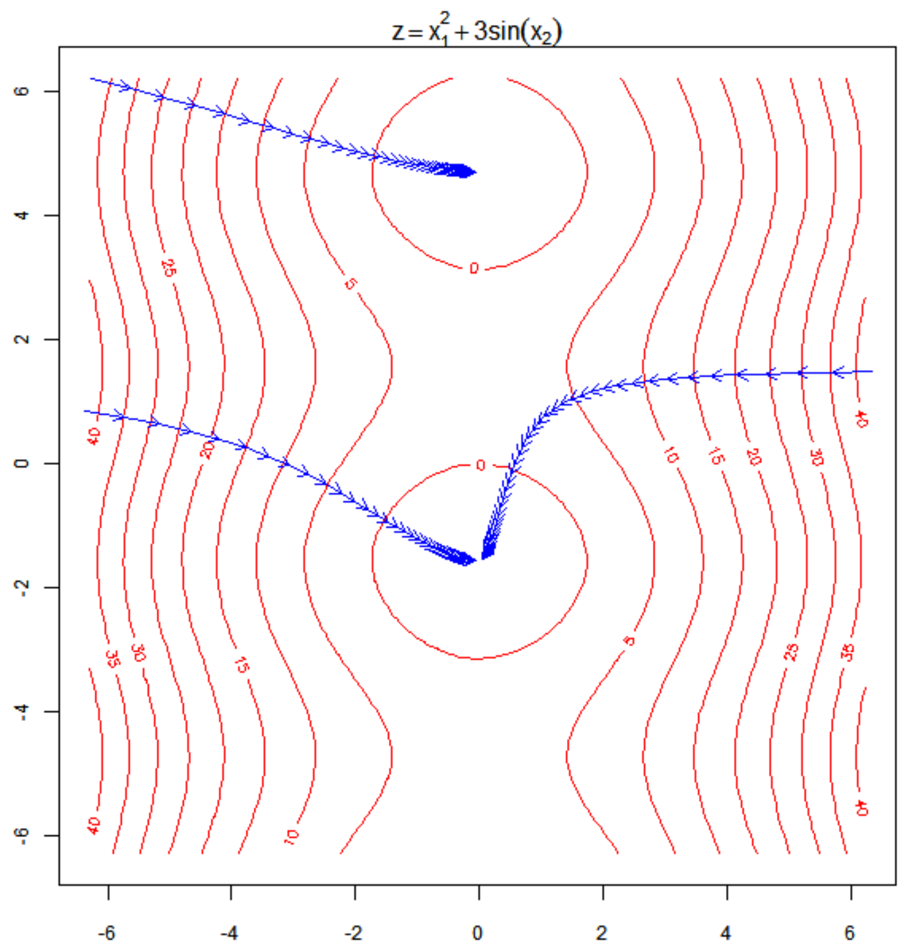


Figure source: yihui.name

Steepest Descent Algorithm

1. Initialize x
2. Repeat
    1. Determine the steepest descent direction Δx
    2. Line search. Choose a step size $t > 0$.
    3. Update. $x := x + t\,\Delta x$.
3. Until stopping criterion is satisfied

What is the Steepest Descent Direction?

Assuming a smooth function, we have that

$$f(x_0 + \Delta x) \approx f(x_0) + \nabla_x f(x_0)^\top \Delta x$$

The (locally at x_0) direction of steepest descent is given by:

$$\begin{aligned}\Delta x^* &= \arg \min_{\Delta x : \|\Delta x\|_2 = 1} f(x_0) + \nabla_x f(x_0)^\top \Delta x \\ &= \arg \min_{\Delta x : \|\Delta x\|_2 = 1} \nabla_x f(x_0)^\top \Delta x\end{aligned}$$

As we have all $a, b \in \mathbb{R}^n$ that $\min_{b:\|b\|_2=1} a^\top b$ is achieved for $b = -\frac{a}{\|a\|_2}$, we have that the steepest descent direction

$$\Delta x^* = -\nabla_x f(x_0)$$

→ Steepest Descent = Gradient Descent

Stepsize Selection: Exact Line Search

$$t = \arg\min_{s \geq 0} f(x + s\Delta x)$$

- Used when the cost of solving the minimization problem with one variable is low compared to the cost of computing the search direction itself.

Stepsize Selection: Backtracking Line Search

- Inexact: step length is chose to approximately minimize f along the ray $\{x + t\,\Delta x \mid t \geq 0\}$

Backtracking Line Search.
given a descent direction Δx for f at $x \in \mathbf{dom} f$, $\alpha \in (0, 0.5)$, $\beta \in (0, 1)$.
$t := 1$
while $f(x + t\Delta x) > f(x) + \alpha t \nabla f(x)^\top \Delta x$, $t := \beta t$.

Stepsize Selection: Backtracking Line Search

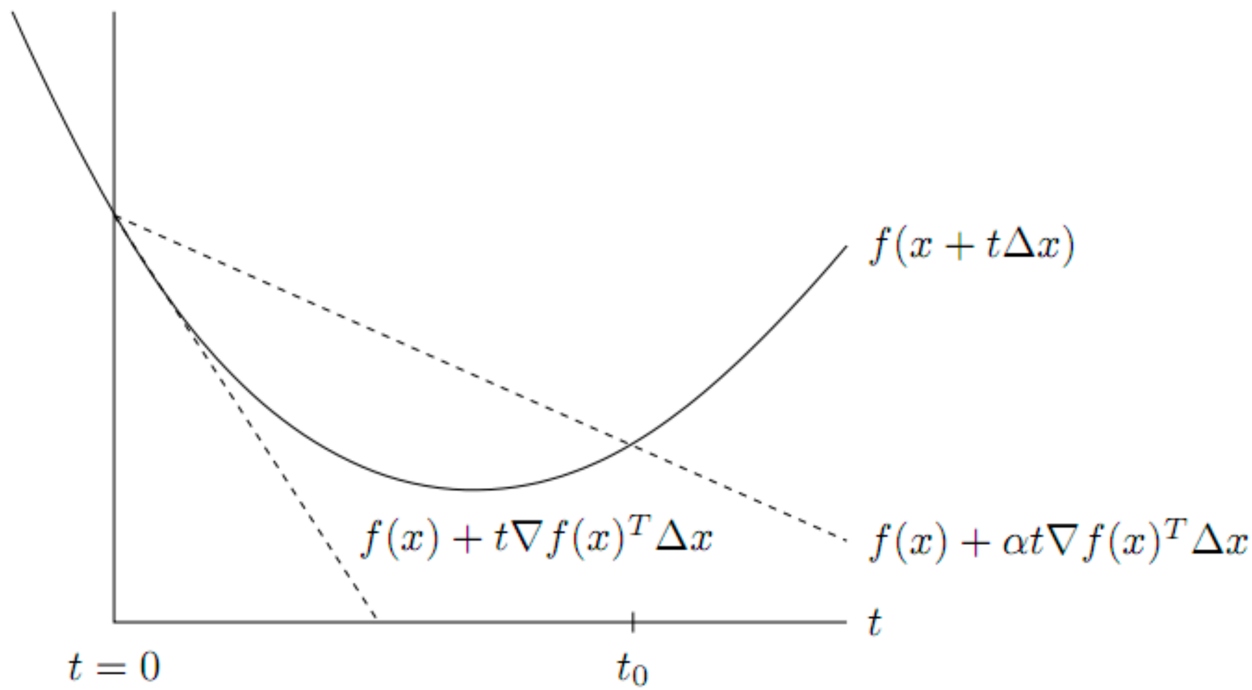


Figure 9.1 *Backtracking line search.* The curve shows f, restricted to the line over which we search. The lower dashed line shows the linear extrapolation of f, and the upper dashed line has a slope a factor of α smaller. The backtracking condition is that f lies below the upper dashed line, *i.e.*, $0 \le t \le t_0$.

Figure source: Boyd and Vandenberghe

Steepest Descent = Gradient Descent

Algorithm 9.3 *Gradient descent method.*

given a starting point $x \in \textbf{dom} f$.

repeat

1. $\Delta x := -\nabla f(x)$.
2. *Line search.* Choose step size t via exact or backtracking line search.
3. *Update.* $x := x + t\Delta x$.

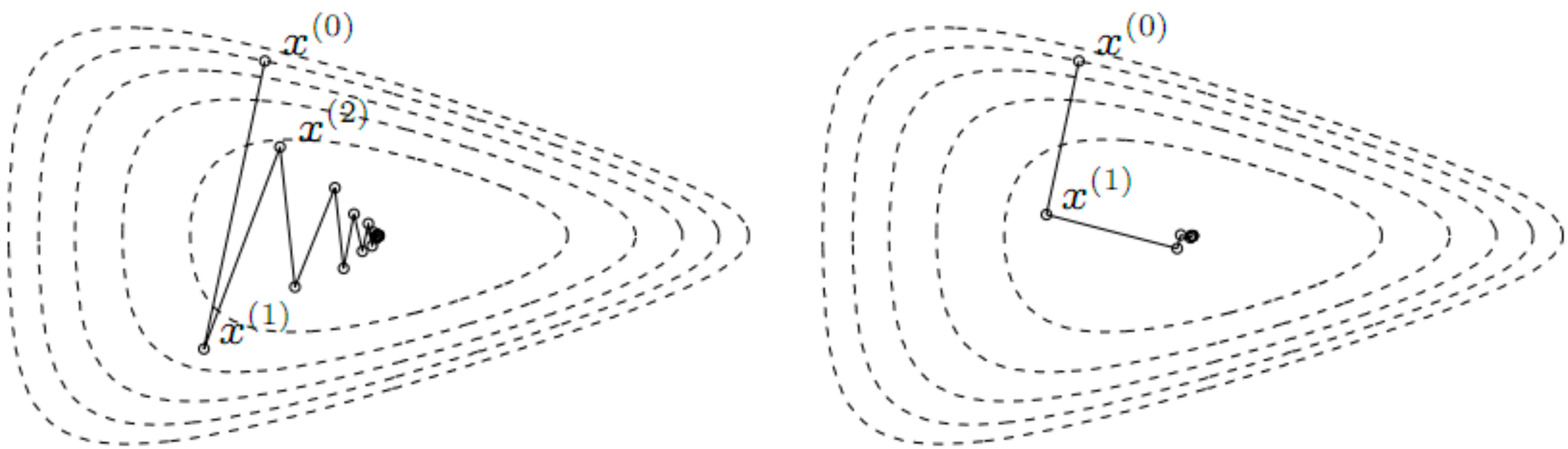
until stopping criterion is satisfied.

The stopping criterion is usually of the form $\|\nabla f(x)\|_2 \leq \eta$, where η is small and positive. In most implementations, this condition is checked after step 1, rather than after the update.

Figure source: Boyd and Vandenberghe

Gradient Descent: Example 1

$$f(x_1, x_2) = e^{x_1+3x_2-0.1} + e^{x_1-3x_2-0.1} + e^{-x_1-0.1}$$



backtracking line search

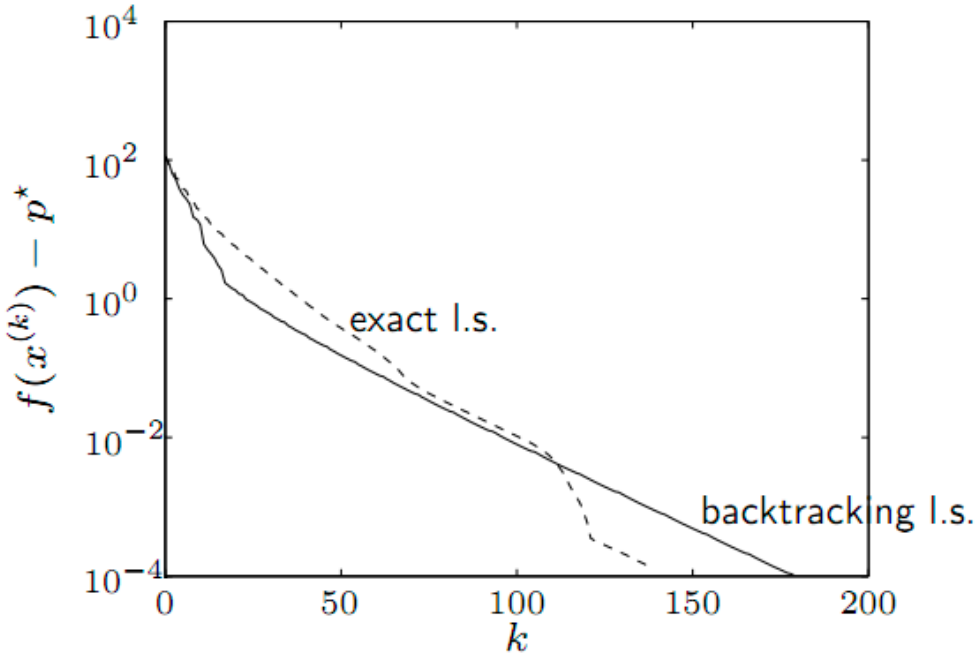
exact line search

Figure source: Boyd and Vandenberghe

Gradient Descent: Example 2

a problem in $\mathbf{R}^{100}$

$$f(x) = c^T x - \sum_{i=1}^{500} \log(b_i - a_i^T x)$$



'linear' convergence, *i.e.*, a straight line on a semilog plot

Figure source: Boyd and Vandenberghe

Gradient Descent: Example 3

$$f(x) = (1/2)(x_1^2 + \gamma x_2^2) \qquad (\gamma > 0)$$

with exact line search, starting at $x^{(0)} = (\gamma, 1)$:

$$x_1^{(k)} = \gamma \left(\frac{\gamma - 1}{\gamma + 1}\right)^k, \qquad x_2^{(k)} = \left(-\frac{\gamma - 1}{\gamma + 1}\right)^k$$

- very slow if $\gamma \gg 1$ or $\gamma \ll 1$
- example for $\gamma = 10$:

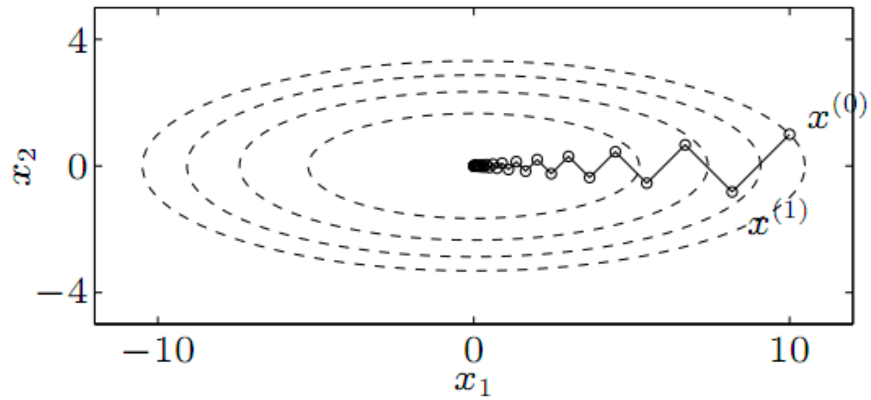
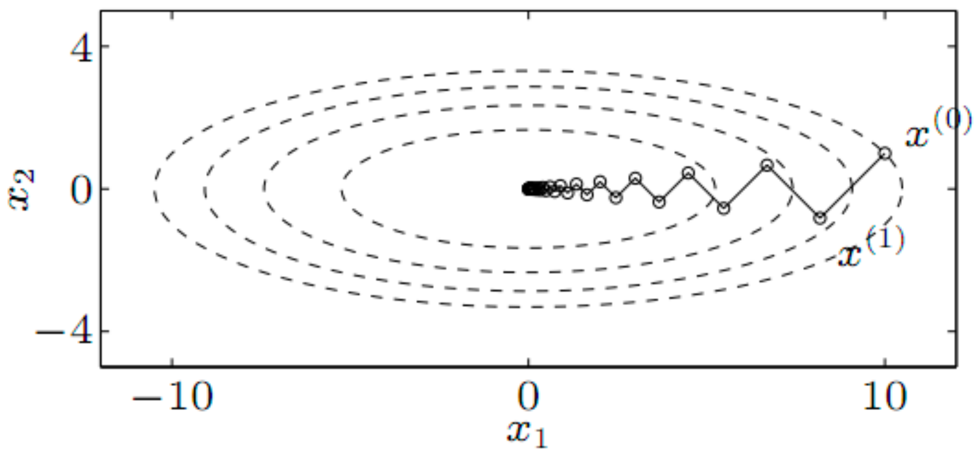
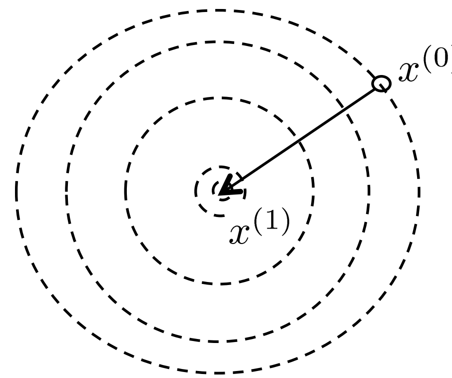


Figure source: Boyd and Vandenberghe

Gradient Descent Convergence



Condition number = 10



Condition number = 1

- For quadratic function, convergence speed depends on ratio of highest second derivative over lowest second derivative ("condition number")
- In high dimensions, almost guaranteed to have a high (=bad) condition number
- Rescaling coordinates (as could happen by simply expressing quantities in different measurement units) results in a different condition number

Outline

- **Unconstrained minimization**
  - Gradient Descent
  - **Newton's Method**
- Equality constrained minimization
- Inequality and equality constrained minimization

Newton's Method (assume f convex for now)

- 2nd order Taylor Approximation rather than 1st order:

$$f(x+\Delta x) \approx f(x) + \nabla f(x)^\top \Delta x + \frac{1}{2}\Delta x^\top \nabla^2 f(x) \Delta x$$

assuming $\nabla^2 f(x) \succeq 0$, the minimum of the 2nd order approximation is achieved at: $\Delta x_{\mathrm{nt}} = -\left(\nabla^2 f(x)\right)^{-1} \nabla f(x)$

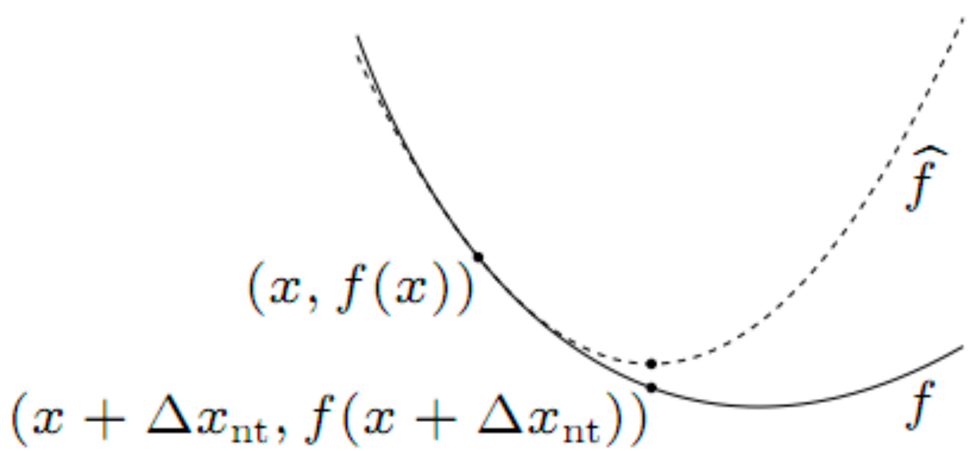


Figure source: Boyd and Vandenberghe

Newton's Method

Algorithm 9.5 *Newton's method.*

given a starting point $x \in \mathbf{dom}\, f$, tolerance $\epsilon > 0$.

repeat

1. *Compute the Newton step and decrement.*
   $\Delta x_{\mathrm{nt}} := -\nabla^2 f(x)^{-1} \nabla f(x); \quad \lambda^2 := \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x).$
2. *Stopping criterion.* **quit** if $\lambda^2/2 \leq \epsilon$.
3. *Line search.* Choose step size t by backtracking line search.
4. *Update.* $x := x + t\Delta x_{\mathrm{nt}}$.

Figure source: Boyd and Vandenberghe

Affine Invariance

- Consider the coordinate transformation $y = A^{-1} x \quad (x = Ay)$

- If running Newton's method starting from $x^{(0)}$ on $f(x)$ results in

  $x^{(0)}, x^{(1)}, x^{(2)}, \ldots$

- Then running Newton's method starting from $y^{(0)} = A^{-1} x^{(0)}$ on $g(y) = f(Ay)$, will result in the sequence

  $y^{(0)} = A^{-1} x^{(0)}, y^{(1)} = A^{-1} x^{(1)}, y^{(2)} = A^{-1} x^{(2)}, \ldots$

- Exercise: try to prove this!

Affine Invariance --- Proof

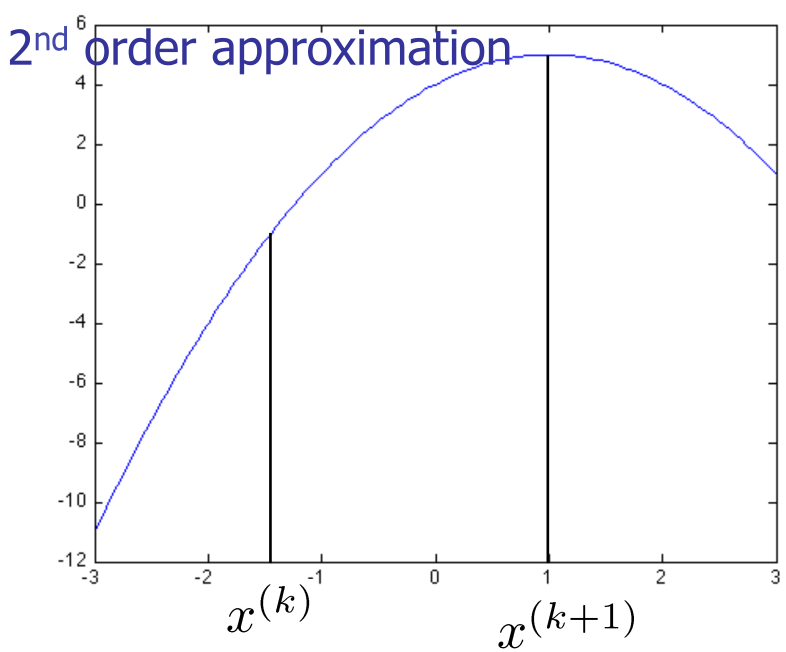
$$\begin{aligned}
\frac{\partial g}{\partial y_i} &= \sum_j \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial y_i} \\
&= \sum_j \frac{\partial f}{\partial x_j} A_{ji} \\
&= (A^\top)_{i,:} \nabla f \\
\nabla g &= A^\top \nabla f
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 g}{\partial y_k \partial y_i} &= \frac{\partial}{\partial y_i}\left(\sum_j \frac{\partial f}{\partial x_j} A_{j,i}\right) \\
&= \sum_j \frac{\partial}{\partial y_k}\left(\frac{\partial f}{\partial x_j}\right) A_{j,i} \\
&= \sum_j \sum_l \frac{\partial^2 f}{\partial x_l \partial x_j}\frac{\partial x_l}{\partial y_k} A_{j,i} \\
&= \sum_j \sum_l \frac{\partial^2 f}{\partial x_l \partial x_j} A_{l,k} A_{j,i} \\
\nabla^2 g &= A^\top \nabla^2 f A
\end{aligned}$$

$$\begin{aligned}
\Delta y &= -\left(\nabla^2 g\right)^{-1} \nabla g \\
&= -\left(A^\top \nabla^2 f A\right)^{-1} A^\top \nabla f \\
&= -A^{-1}\left(\nabla^2 f\right)^{-1} A^{-\top} A^\top \nabla f \\
&= -A^{-1}\left(\nabla^2 f\right)^{-1} \nabla f \\
&= A^{-1}\Delta x
\end{aligned}$$

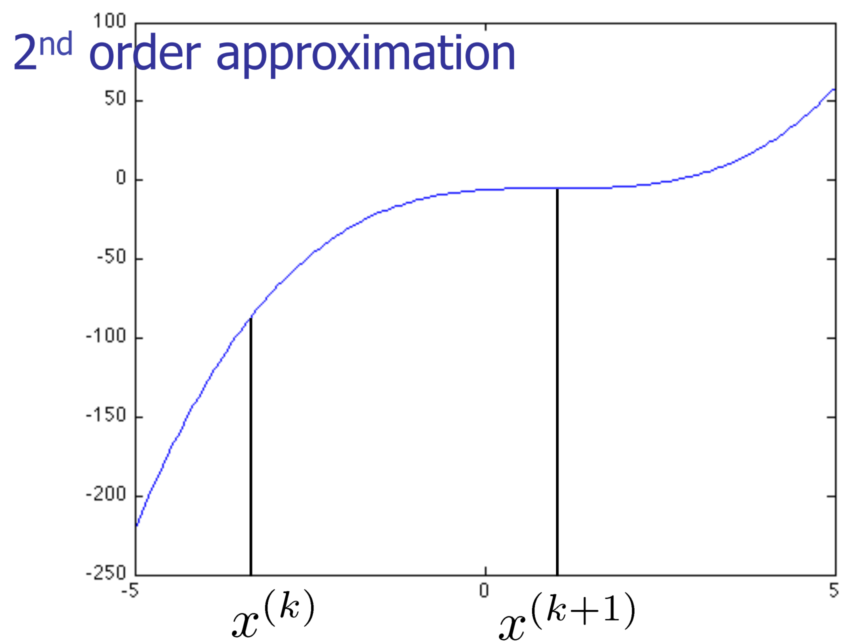
Newton's method when f not convex (i.e. not $\nabla^2 f(x) \succeq 0$)

- Example 1:



→ ended up at max rather than min !

- Example 2:



→ ended up at inflection point rather than min !

Newton's method when f not convex (i.e. not $\nabla^2 f(x) \succeq 0$)

- Issue: now Δx_{nt} does not lead to the local minimum of the quadratic approximation --- it simply leads to the point where the gradient of the quadratic approximation is zero, this could be a maximum or a saddle point

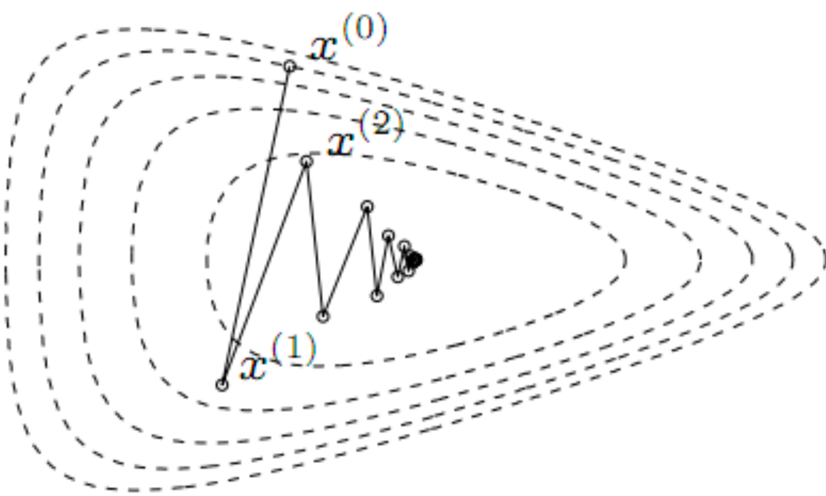
- Possible fixes, let $X \Lambda X^\top = \nabla^2 f(x)$ be the eigenvalue decomposition.
  - Fix 1: Replace $\nabla^2 f(x)$ with $X \bar{\Lambda} X^\top$, with $\bar{\Lambda}$ a diagonal matrix with $\bar{\Lambda}_{i,i} = \max(0, \Lambda_{i,i})$.
  - Fix 2: Replace $\nabla^2 f(x)$ with $X \bar{\Lambda} X^\top$, with $\bar{\Lambda}$ a diagonal matrix with $\bar{\Lambda}_{i,i} = \Lambda_{i,i} + (-1) * \min_j \Lambda_{j,j}$
  - Fix 3: Replace original objective $f(x)$ with $f(x) + \nu \|\Delta x\|_2^2$ with ν greater than $-\min_j \Lambda_{j,j}$

    ("proximal method")
  - Fix 4: Use a gradient descent step, rather than a Newton step, in the current iteration.

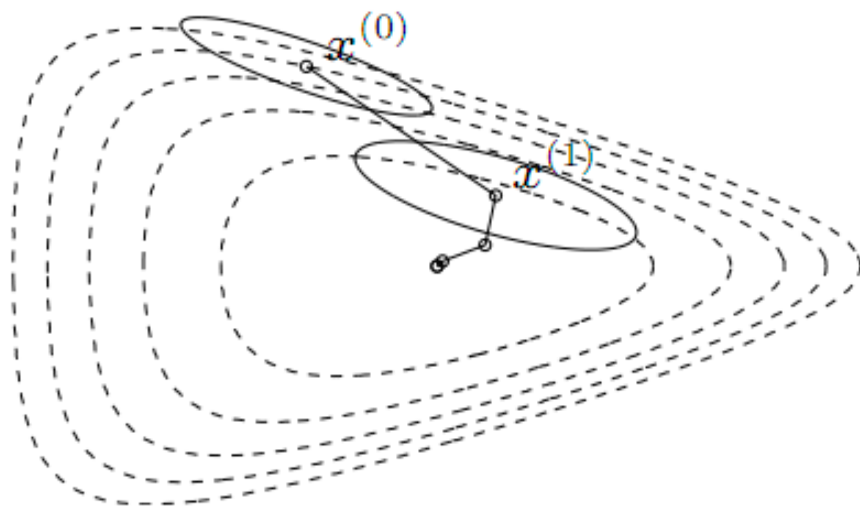
In my experience Fix 3 works best.

Example 1

$$f(x_1, x_2) = e^{x_1+3x_2-0.1} + e^{x_1-3x_2-0.1} + e^{-x_1-0.1}$$



gradient descent with
backtracking line search



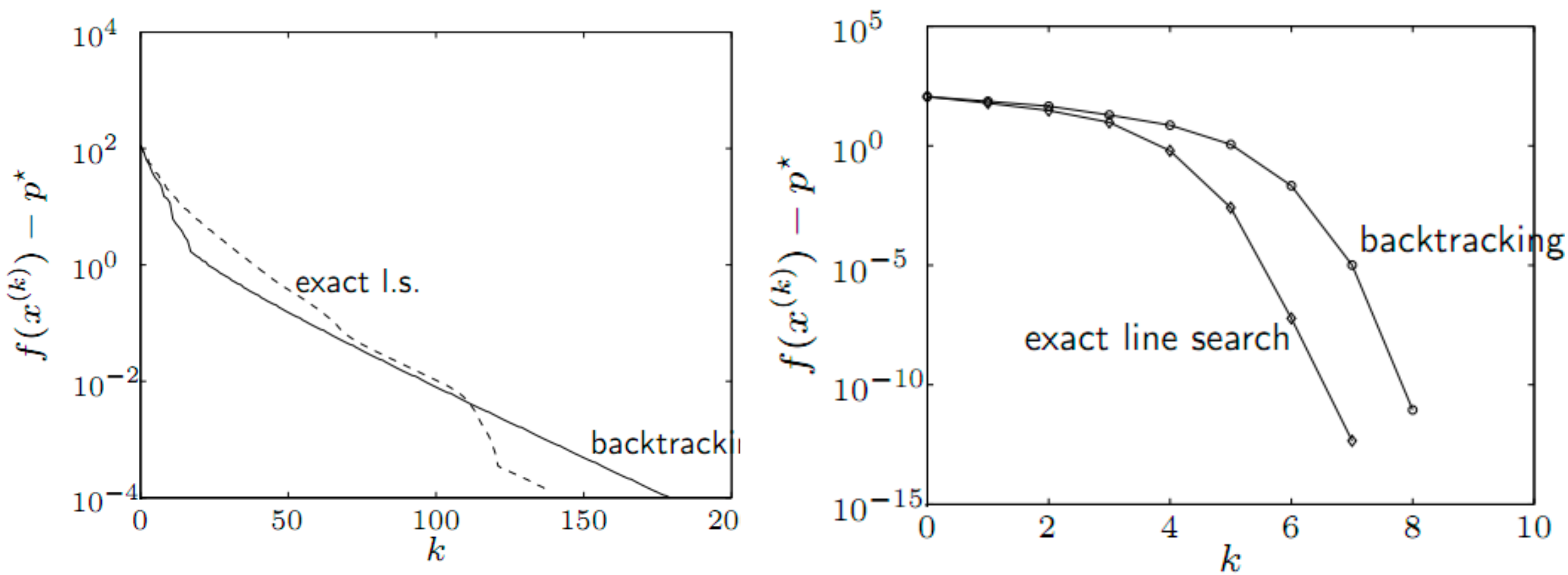
Newton's method with
backtracking line search

Figure source: Boyd and Vandenberghe

Example 2

a problem in R^{100}

$$f(x) = c^T x - \sum_{i=1}^{500} \log(b_i - a_i^T x)$$



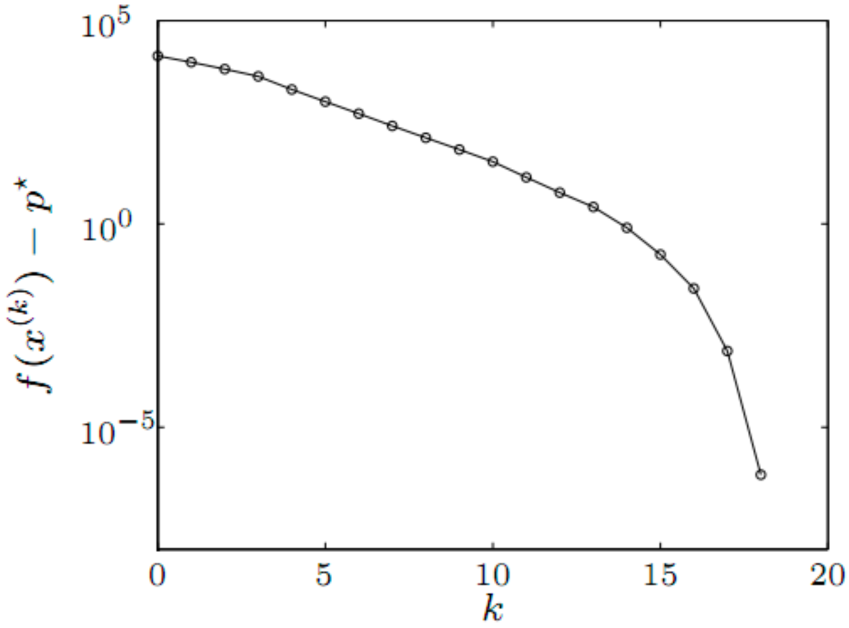
gradient descent **Newton's method**

Figure source: Boyd and Vandenberghe

Larger Version of Example 2

example in $\mathbf{R}^{10000}$ (with sparse a_i)

$$f(x) = -\sum_{i=1}^{10000} \log(1 - x_i^2) - \sum_{i=1}^{100000} \log(b_i - a_i^T x)$$



- backtracking parameters $\alpha = 0.01$, $\beta = 0.5$.
- performance similar as for small examples

Gradient Descent: Example 3

$$f(x) = (1/2)(x_1^2 + \gamma x_2^2) \qquad (\gamma > 0)$$

with exact line search, starting at $x^{(0)} = (\gamma, 1)$:

$$x_1^{(k)} = \gamma \left(\frac{\gamma - 1}{\gamma + 1}\right)^k, \qquad x_2^{(k)} = \left(-\frac{\gamma - 1}{\gamma + 1}\right)^k$$

- very slow if $\gamma \gg 1$ or $\gamma \ll 1$
- example for $\gamma = 10$:

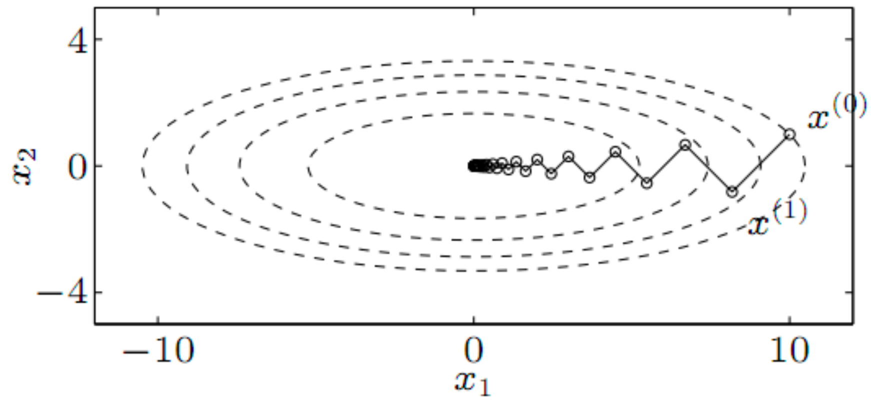
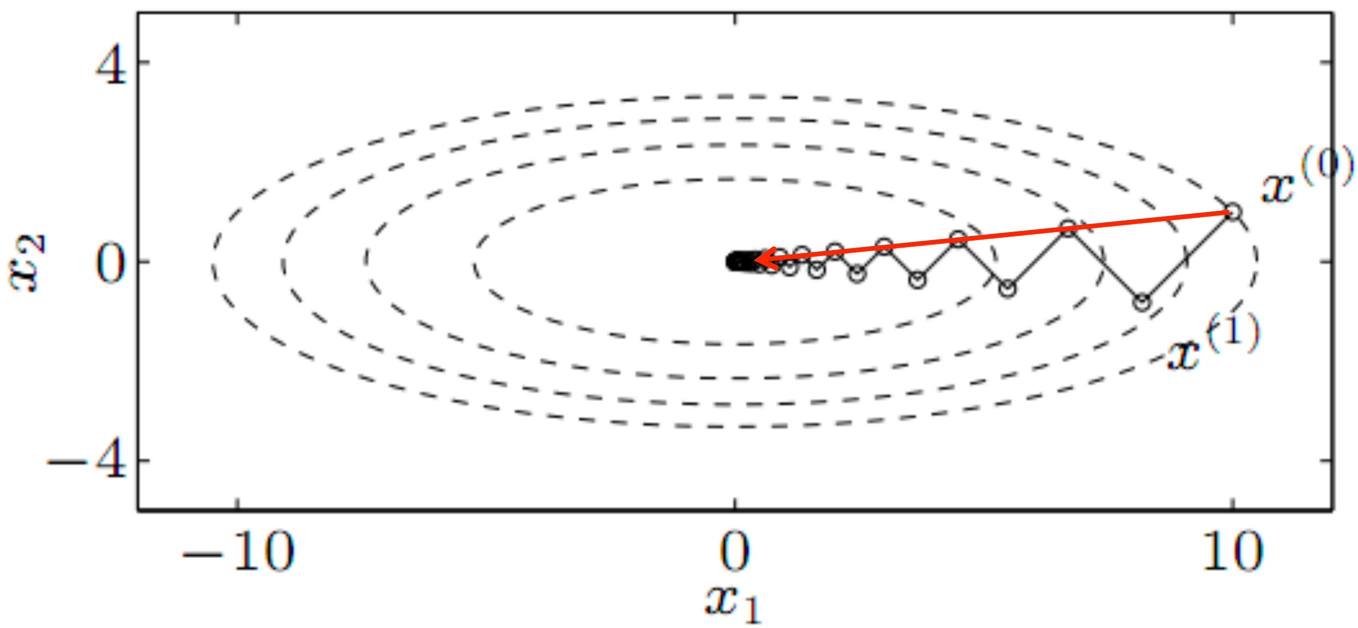


Figure source: Boyd and Vandenberghe

Example 3



- Gradient descent
- Newton's method (converges in one step if f convex quadratic)

Quasi-Newton Methods

- Quasi-Newton methods use an approximation of the Hessian
  - Example 1: Only compute diagonal entries of Hessian, set others equal to zero. Note this also simplifies computations done with the Hessian.
  - Example 2: natural gradient --- see next slide

Natural Gradient

- Consider a standard maximum likelihood problem:

$$\max_\theta f(\theta) = \max_\theta \sum_i \log p(x^{(i)};\theta)$$

- Gradient: $$\frac{\partial f(\theta)}{\partial \theta_p} = \sum_i \frac{\partial \log p(x^{(i)};\theta)}{\partial \theta_p} = \sum_i \frac{\partial p(x^{(i)};\theta)}{\partial \theta_p} \frac{1}{p(x^{(i)};\theta)}$$

- Hessian:

$$\frac{\partial^2 f(\theta)}{\partial \theta_q \partial \theta_p} = \sum_i \frac{\partial^2 p(x^{(i)};\theta)}{\partial \theta_q \partial \theta_p} \frac{1}{p(x^{(i)};\theta)} - \frac{\partial p(x^{(i)};\theta)}{\partial \theta_q} \frac{1}{p(x^{(i)};\theta)} \frac{\partial p(x^{(i)};\theta)}{\partial \theta_p} \frac{1}{p(x^{(i)};\theta)}$$

$$\nabla^2 \log f(\theta) = \sum_i \frac{\nabla^2 p(x^{(i)};\theta)}{p(x^{(i)};\theta)} - \left(\nabla \log p(x^{(i)};\theta)\right) \left(\nabla \log p(x^{(i)};\theta)\right)^\top$$

- Natural gradient:

$$= \left(\sum_i \left(\nabla \log p(x^{(i)};\theta)\right) \left(\nabla \log p(x^{(i)};\theta)\right)^\top\right)^{-1} \left(\sum_i \nabla \log p(x^{(i)};\theta)\right)$$

only keeps the 2nd term in the Hessian. Benefits: (1) faster to compute (only gradients needed); (2) guaranteed to be negative definite; (3) found to be superior in some experiments; (4) invariant to re-parametrization

Natural Gradient

- Property: Natural gradient is invariant to parameterization of the family of probability distributions p(x ; θ)

- Hence the name.

- Note this property is stronger than the property of Newton's method, which is invariant to affine re-parameterizations only.

- Exercise: Try to prove this property!

Natural Gradient Invariant to Reparametrization --- Proof

- Natural gradient for parametrization with θ:

$$\bar{g}_\theta = \left(\sum_i \left(\nabla_\theta \log p(x^{(i)};\theta)\right)\left(\nabla_\theta \log p(x^{(i)};\theta)\right)^\top\right)^{-1}\left(\sum_i \nabla_\theta \log p(x^{(i)};\theta)\right)$$

- Let ϕ = f(θ), and let $J = \frac{\partial\theta}{\partial\phi}$ i.e., $J_{i,j} = \frac{\partial\theta_i}{\partial\phi_j}$

$$\begin{aligned}
\bar{g}_\phi &= \left(\sum_i \left(\nabla_\phi \log p(x^{(i)};\phi)\right)\left(\nabla_\phi \log p(x^{(i)};\phi)\right)^\top\right)^{-1}\left(\sum_i \nabla_\phi \log p(x^{(i)};\phi)\right) \\
&= \left(\sum_i \left(J^\top\nabla_\theta \log p(x^{(i)};\phi)\right)\left(J^\top\nabla_\theta \log p(x^{(i)};\phi)\right)^\top\right)^{-1}\left(J^\top\sum_i \nabla_\theta \log p(x^{(i)};\phi)\right) \\
&= J^\top\bar{g}_\theta
\end{aligned}$$

→ the natural gradient direction is the same independent of the (invertible, but otherwise not constrained) reparametrization f

Outline

- Unconstrained minimization
  - Gradient Descent
  - Newton's Method
- **Equality constrained minimization**
- Inequality and equality constrained minimization

Equality Constrained Minimization

- Problem to be solved:

$$
\begin{aligned}
\min_x \quad & f(x) \\
\text{s.t.} \quad & Ax = b
\end{aligned}
$$

- We will cover three solution methods:
  - Elimination
  - Newton's method
  - Infeasible start Newton method

Method 1: Elimination

- From linear algebra we know that there exist a matrix F (in fact infinitely many) such that:

$$\{x|Ax = b\} = \{x|x = \hat{x} + Fz\}$$

$\hat{x}$ can be any solution to Ax = b

F spans the nullspace of A

A way to find an F: compute SVD of A, A = U S V', for A having k nonzero singular values, set F = U(:, k+1:end)

- So we can solve the equality constrained minimization problem by solving an ***unconstrained minimization problem over a new variable z***:

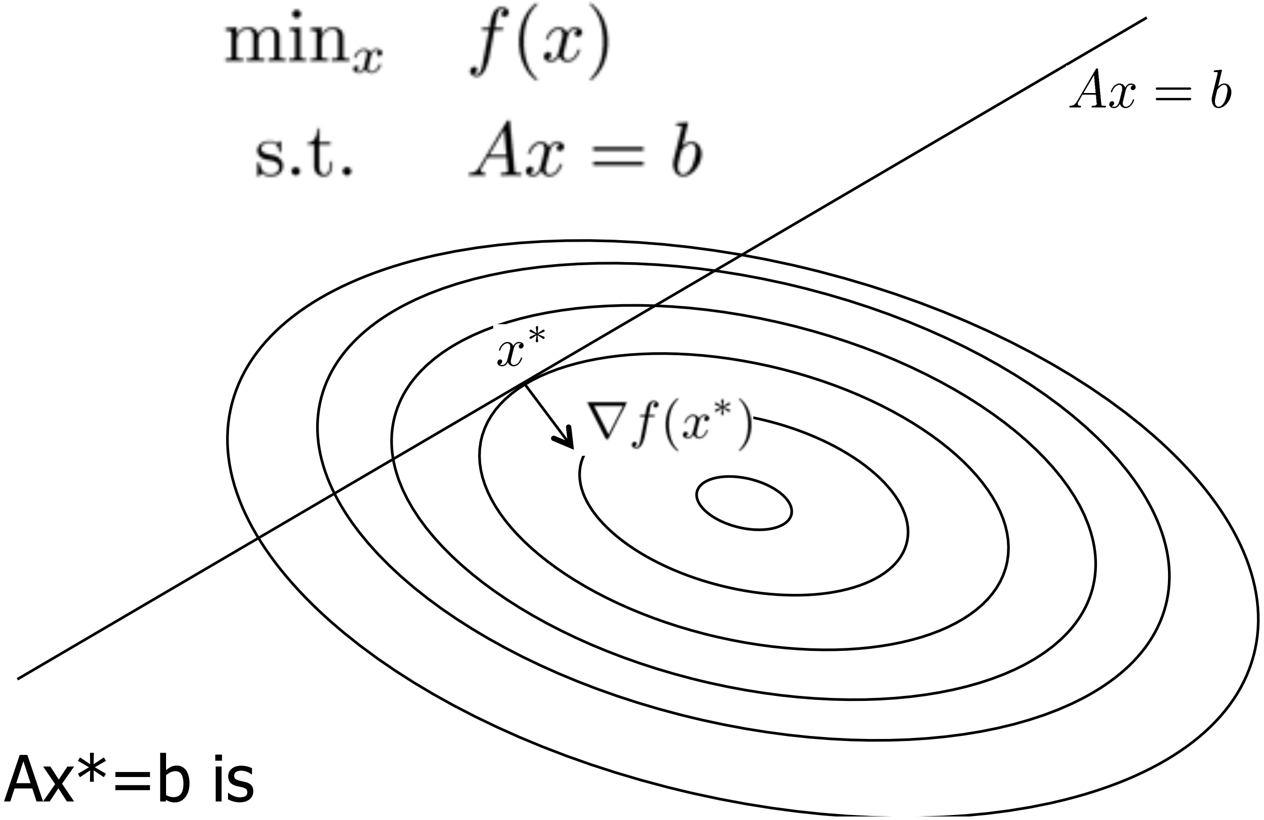
$$\min_z f(\hat{x} + Fz)$$

- Potential cons: (i) need to first find a solution to Ax=b, (ii) need to find F, (iii) elimination might destroy sparsity in original problem structure

Methods 2 and 3 Require Us to First Understand the Optimality Condition

- Recall problem to be solved:

$$\min_x \quad f(x)$$
$$\text{s.t.} \quad Ax = b$$



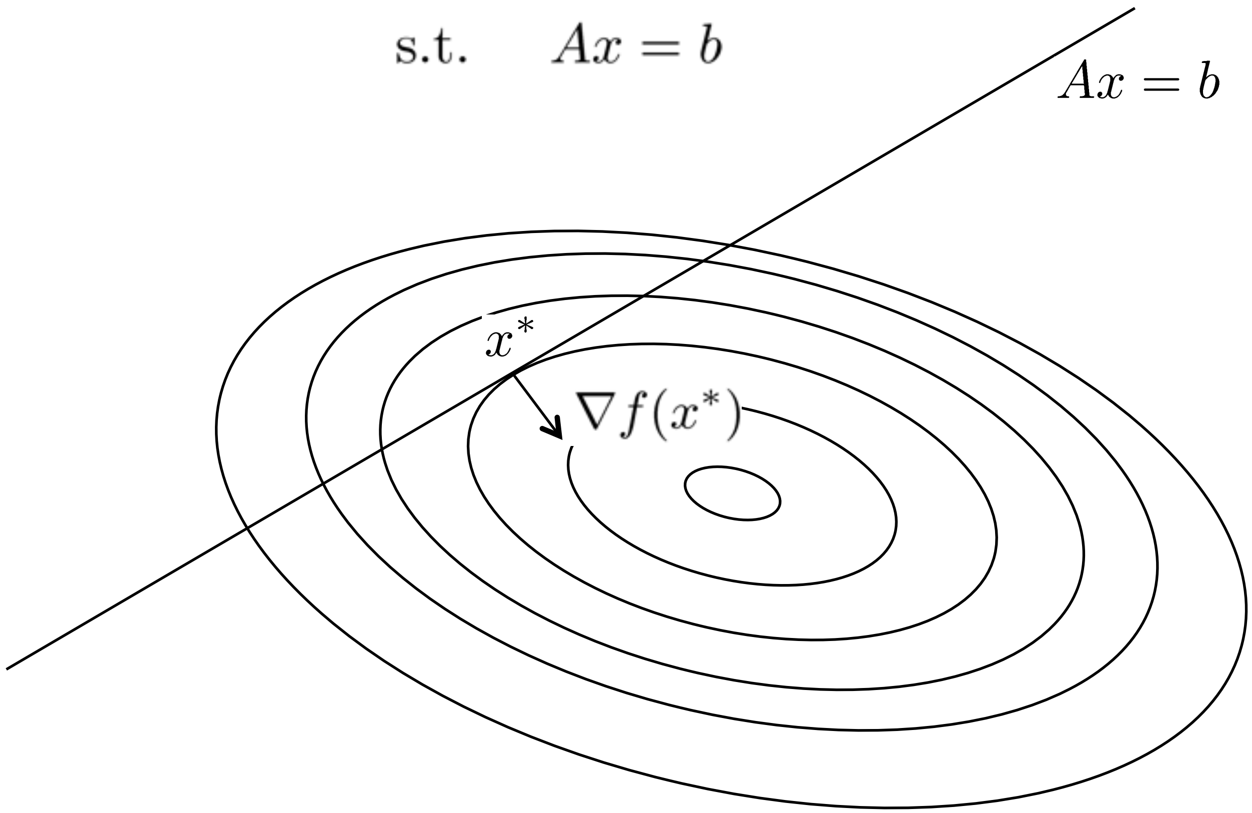
x* with Ax*=b is (local) optimum iff: $\forall \Delta x \quad \text{if } A\Delta x = 0 \text{ then } \nabla f(x^*)^\top \Delta x = 0.$

Equivalently: $\nabla f(x^*)^\top = \nu^\top A$

Methods 2 and 3 Require Us to First Understand the Optimality Condition

- Recall the problem to be solved:

$$\begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & Ax = b \end{array}$$



Optimality Condition: $Ax^* = b$ and $\nabla f(x^*) + A^\top \nu = 0$

Method 2: Newton's Method

- Problem to be solved: $$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$
- Optimality Condition: $Ax^* = b$ and $\nabla f(x^*) + A^\top \nu = 0$
- Assume x is feasible, i.e., satisfies Ax = b, now use 2nd order approximation of f:

$$\begin{aligned} \min_{\Delta x} \quad & f(x) + \nabla f(x)^\top \Delta x + \frac{1}{2} \Delta x^\top \nabla^2 f(x) \Delta x \\ \text{s.t.} \quad & A(x + \Delta x) = b \end{aligned}$$

- → Optimality condition for 2nd order approximation:

$$\begin{bmatrix} \nabla^2 f(x) & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \nu \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

Method 2: Newton's Method

---

given starting point $x \in \textbf{dom}\, f$ with $Ax = b$, tolerance $\epsilon > 0$.

repeat

1. Compute the Newton step and decrement $\Delta x_{\rm nt}$, $\lambda(x)$.
2. *Stopping criterion.* **quit** if $\lambda^2/2 \le \epsilon$.
3. *Line search.* Choose step size t by backtracking line search.
4. *Update.* $x := x + t\Delta x_{\rm nt}$.

With Newton step obtained by solving a linear system of equations:

$$\begin{bmatrix} \nabla^2 f(x) & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\rm nt} \\ \nu \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

Feasible descent method: $x^{(k)}$ feasible and $f(x^{(k+1)}) \le f(x^{(k)})$

Method 3: Infeasible Start Newton Method

- Problem to be solved:
$$\begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & Ax = b \end{array}$$

- Optimality Condition: $Ax^* = b$ and $\nabla f(x^*) + A^\top \nu = 0$

- Use 1st order approximation of the optimality conditions at current x:
$$\begin{aligned} A(x + \Delta x) &= b \\ \nabla f(x) + \nabla^2 f(x)\Delta x + A^\top \nu &= 0 \end{aligned}$$

$$\begin{bmatrix} \nabla^2 f(x) & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \nu \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ b - Ax \end{bmatrix}$$

Optimal Control

- We can now solve:

$$\begin{aligned} \min_{x,u} \quad & \sum_{t=0}^{T} g_t(x_t, u_t) \\ \text{s.t.} \quad & x_{t+1} = A_t x_t + B_t u_t \quad \forall t \end{aligned}$$

- And often one can efficiently solve

$$\begin{aligned} \min_{x,u} \quad & \sum_{t=0}^{T} g_t(x_t, u_t) \\ \text{s.t.} \quad & x_{t+1} = f_t(x_t, u_t) \quad \forall t \end{aligned}$$

by iterating over (i) linearizing the constraints, and (ii) solving the resulting problem.

Optimal Control: A Complete Algorithm

- Given: $\bar{x}_0$
- For k=0, 1, 2, …, T
  - Solve

$$\begin{aligned} \min_{x,u} \quad & \sum_{t=k}^{T} g_t(x_t, u_t) \\ \text{s.t.} \quad & x_{t+1} = f_t(x_t, u_t) \quad \forall t \in \{k, k+1, \ldots, T-1\} \\ & x_k = \bar{x}_k \end{aligned}$$

  - Execute u_k
  - Observe resulting state, $\bar{x}_{k+1}$

→ = an instantiation of Model Predictive Control.

→ Initialization with solution from iteration k-1 can make solver very fast (and would be done most conveniently with infeasible start Newton method)

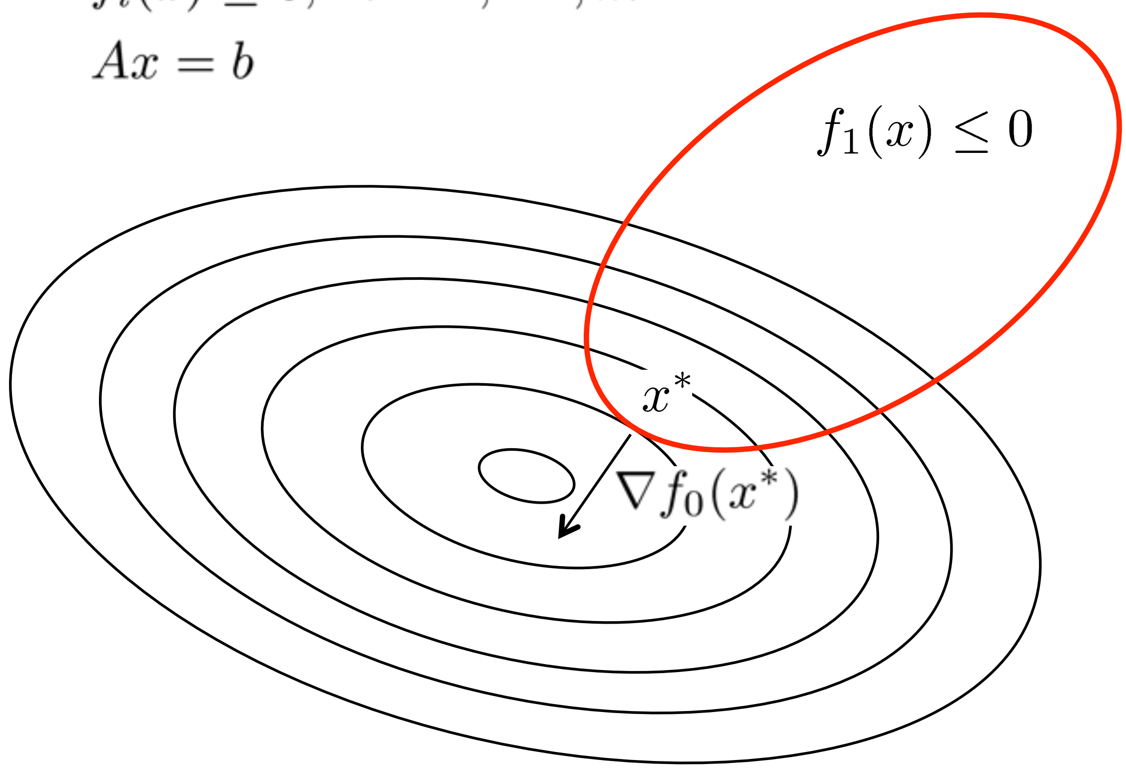
Outline

- Unconstrained minimization
- Equality constrained minimization
- **Inequality and equality constrained minimization**

Equality and Inequality Constrained Minimization

- Recall the problem to be solved:

$$\begin{aligned} \min_x \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{aligned}$$



Equality and Inequality Constrained Minimization

- Problem to be solved:

$$\begin{array}{ll} \min_x & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- Reformulation via indicator function,

$$\begin{array}{ll} \min_x & f_0(x) + \sum_{I=1}^{m} I_-(f_i(x)) \\ & Ax = b \end{array}$$

$$I_-(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ \infty & \text{otherwise} \end{cases}$$

→ No inequality constraints anymore, but very poorly conditioned objective function

Equality and Inequality Constrained Minimization

- Problem to be solved:

$$\begin{aligned} \min_x \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

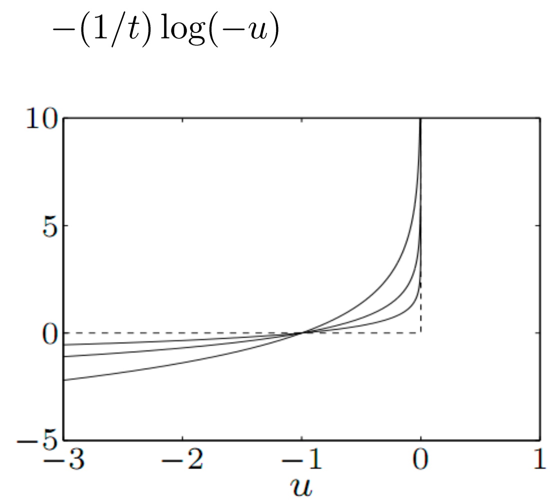
- Reformulation via indicator function

$$\begin{aligned} \min_x \quad & f_0(x) + \sum_{I=1}^{m} I_-(f_i(x)) \\ & Ax = b \end{aligned}$$

→ No inequality constraints anymore, but very poorly conditioned objective function

- Approximation via logarithmic barrier:

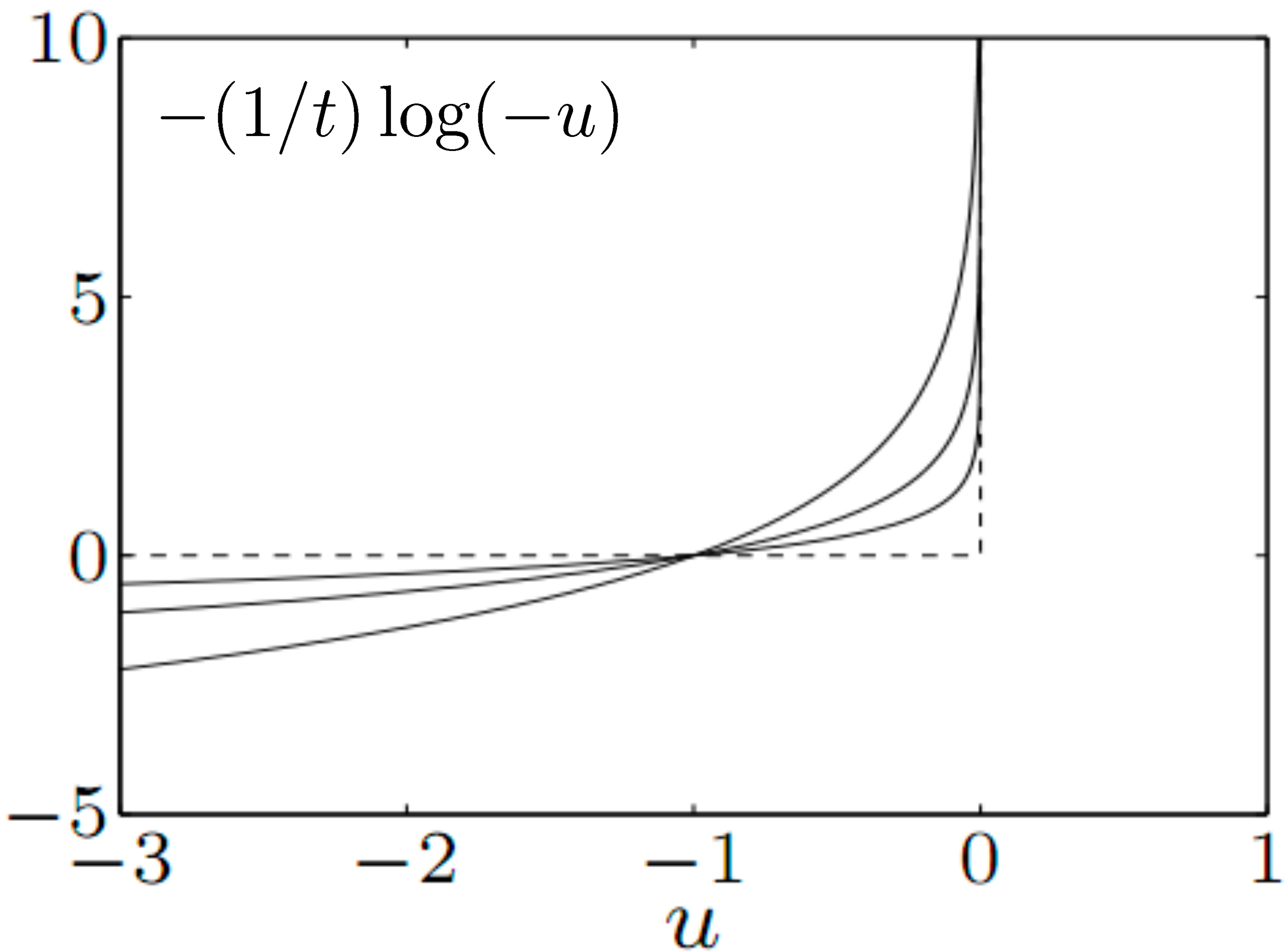
$$\begin{aligned} \min_x \quad & f_0(x) - (1/t) \sum_{i=1}^{m} \log(-f_i(x)) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$



for t>0, -(1/t) log(-u) is a smooth approximation of *I_*(u)

approximation improves for *t* → ∞, better conditioned for smaller *t*

Equality and Inequality Constrained Minimization



Barrier Method

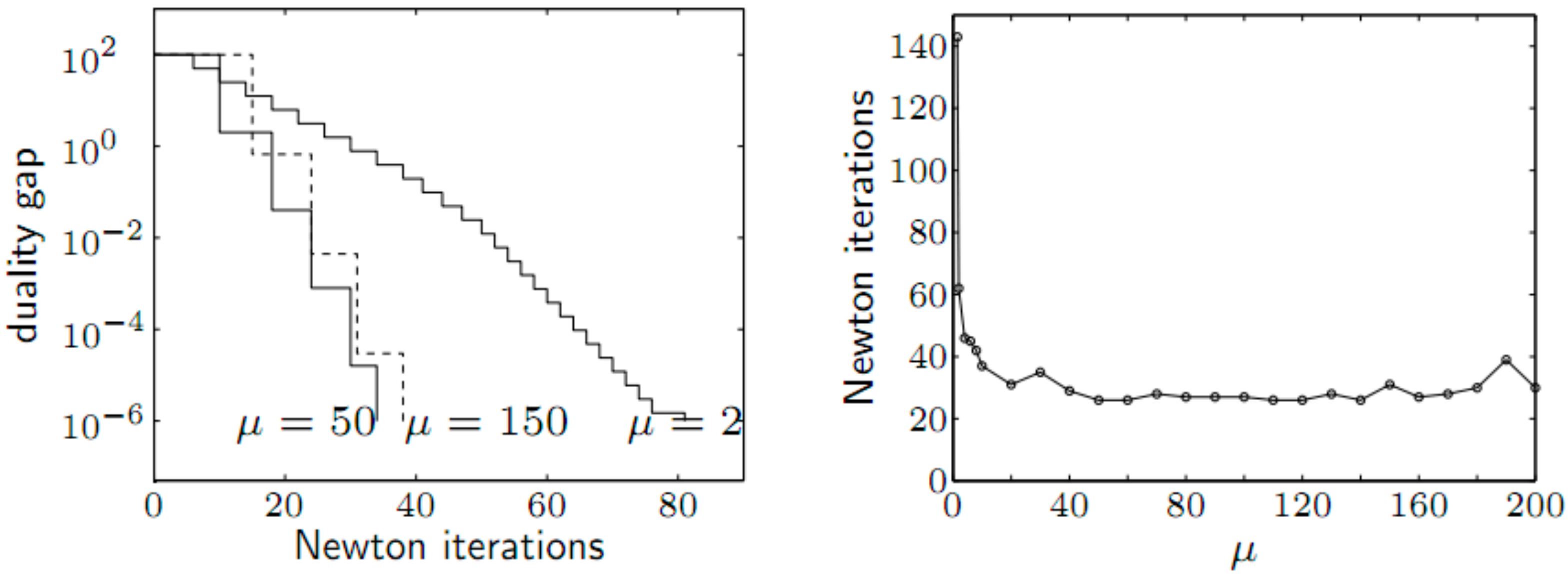
- Given: strictly feasible x, $t=t^{(0)} > 0$, $\mu > 1$, tolerance $\epsilon > 0$
- Repeat
  1. *Centering Step.* Compute $x^*(t)$ by solving

$$
\begin{aligned}
\min_x \quad & f_0(x) - (1/t)\sum_{i=1}^{m} \log(-f_i(x)) \\
\text{s.t.} \quad & Ax = b
\end{aligned}
$$

     starting from x
  2. *Update.* $x := x^*(t)$.
  3. *Stopping Criterion.* Quit if $m/t < \epsilon$
  4. *Increase t.* $t := \mu t$

Example 1: Inequality Form LP

inequality form LP ($m = 100$ inequalities, $n = 50$ variables)

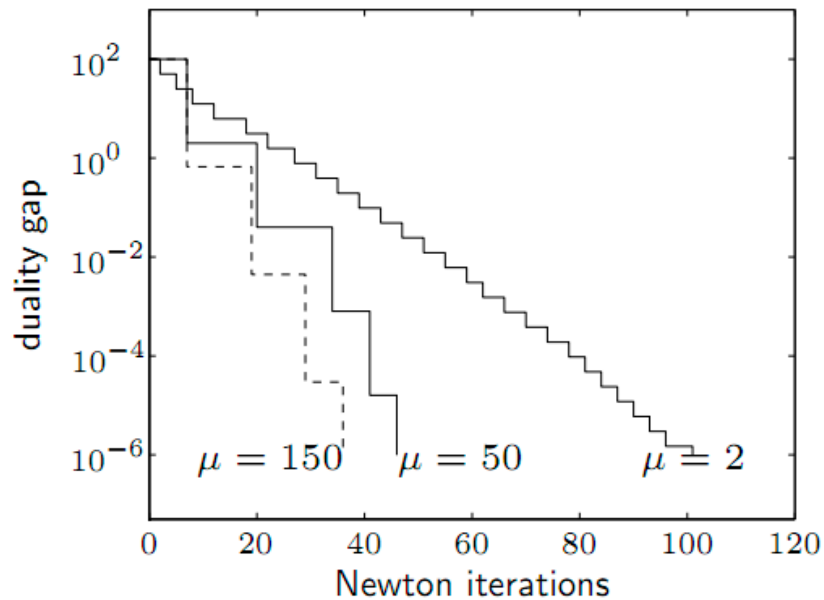


- starts with x on central path ($t^{(0)} = 1$, duality gap 100)
- terminates when $t = 10^8$ (gap 10^{-6})
- centering uses Newton's method with backtracking
- total number of Newton iterations not very sensitive for $\mu \geq 10$

Example 2: Geometric Program

geometric program ($m = 100$ inequalities and $n = 50$ variables)

$$\begin{array}{ll} \text{minimize} & \log\left(\sum_{k=1}^{5} \exp(a_{0k}^T x + b_{0k})\right) \\ \text{subject to} & \log\left(\sum_{k=1}^{5} \exp(a_{ik}^T x + b_{ik})\right) \leq 0, \quad i = 1, \ldots, m \end{array}$$

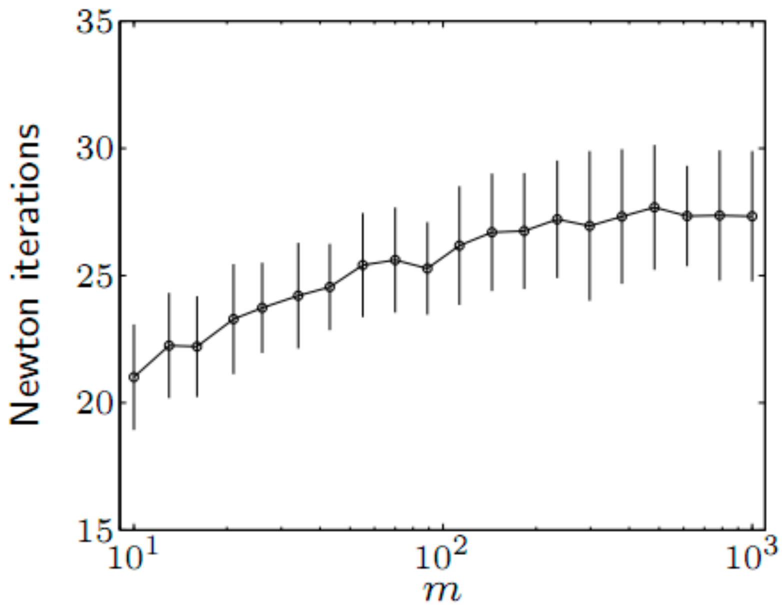


Example 3: Standard LPs

family of standard LPs $(A \in \mathbf{R}^{m \times 2m})$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \succeq 0 \end{array}$$

$m = 10, \ldots, 1000$; for each m, solve 100 randomly generated instances



number of iterations grows very slowly as m ranges over a $100 : 1$ ratio

Initalization

- Basic phase I method:

Initialize by first solving:

$$\begin{aligned} \min_{x,s} \quad & s \\ \text{s.t.} \quad & f_i(x) \le s, \quad i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

- Easy to initialize above problem, pick some x such that Ax = b, and then simply set $s = \max_i f_i(x)$
- Can stop early---whenever s < 0

Initalization

- Sum of infeasibilities phase I method:
- Initialize by first solving:

$$
\begin{aligned}
\min_{x,s} \quad & \sum_{I=1}^{m} s_i \\
\text{s.t.} \quad & f_i(x) \leq s_i, \quad i = 1, \ldots, m \\
& s_i \geq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{aligned}
$$

- Easy to initialize above problem, pick some x such that Ax = b, and then simply set $s_i = \max(0, f_i(x))$
- For infeasible problems, produces a solution that satisfies many more inequalities than basic phase I method

Other methods

- We have covered a primal interior point method
  - one of several optimization approaches
- Examples of others:
  - Primal-dual interior point methods
  - Primal-dual infeasible interior point methods

Optimal Control

- We can now solve:

$$\begin{aligned} \min_{x,u} \quad & \sum_{t=0}^{T} g_t(x_t, u_t) \\ \text{s.t.} \quad & x_{t+1} = A_t x_t + B_t u_t \quad \forall t \\ & f_i(x, u) \leq 0, \quad i = 1, \ldots, m \end{aligned}$$

- And often one can efficiently solve

$$\begin{aligned} \min_{x,u} \quad & \sum_{t=0}^{T} g_t(x_t, u_t) \\ \text{s.t.} \quad & x_{t+1} = f_t(x_t, u_t) \quad \forall t \\ & \hat{f}_i(x, u) \leq 0, \quad i = 1, \ldots, m \end{aligned}$$

by iterating over (i) linearizing the equality constraints, convexly approximating the inequality constraints with convex inequality constraints, and (ii) solving the resulting problem.

CVX

- Disciplined convex programming
  - = convex optimization problems of forms that it can easily verify to be convex
- Convenient high-level expressions
- Excellent for fast implementation

- Designed by Michael Grant and Stephen Boyd, with input from Yinyu Ye.
- Current webpage: http://cvxr.com/cvx/

CVX

- Matlab Example for Optimal Control, see course webpage