Function Approximation

Pieter Abbeel
UC Berkeley EECS

Outline

- Value iteration with function approximation
- Linear programming with function approximation

Value Iteration

- Algorithm:
  - Start with $V_0^*(s) = 0$ for all s.
  - For i=1, … , H

    For all states s $\in$ S:

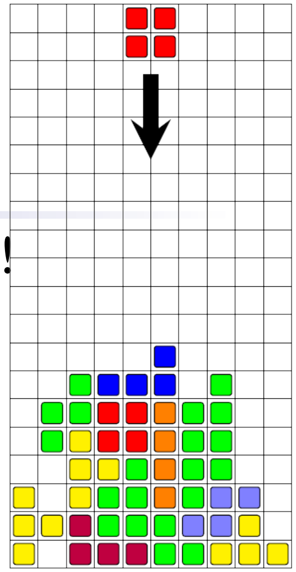
$$V_{i+1}^*(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i^*(s')\right]$$

$$\pi_{i+1}^*(s) \leftarrow \arg\max_{a \in A} \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V_i^*(s')]$$

Impractical for large state spaces

- $V_i^*(s)$ = the expected sum of rewards accumulated when starting from state s and acting optimally for a horizon of i steps
- $\pi_i^*(s)$ = the optimal action when in state s and getting to act for a horizon of i steps

Example: tetris

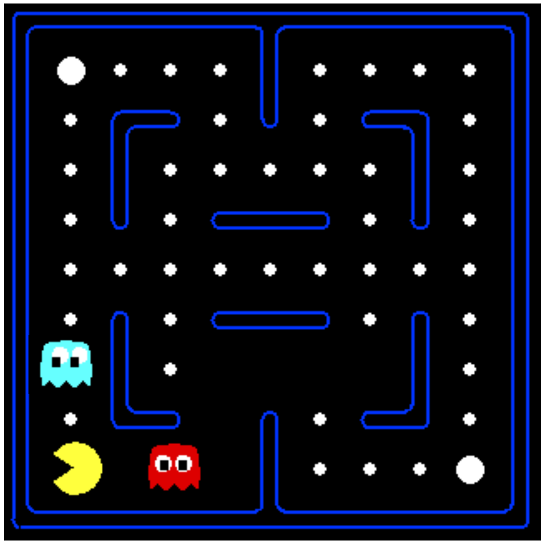


- state: board configuration + shape of the falling piece ~2^{200} states!
- action: rotation and translation applied to the falling piece
- 22 features aka basis functions ϕ_i
  - Ten basis functions, 0, . . . , *9, mapping the state to the height h[k] of each of the ten* columns.
  - Nine basis functions, 10*, . . . , 18, each mapping the state to the absolute difference* between heights of successive columns: |*h[k+1] – h[k]*|*, k = 1, . . . , 9.*
  - One basis function, 19, that maps state to the maximum column height: $\max_k h[k]$
  - One basis function, 20, that maps state to the number of 'holes' in the board.
  - One basis function, 21, that is equal to 1 in every state.

$$\hat{V}_\theta(s) = \sum_{i=0}^{21} \theta_i \phi_i(s) = \theta^\top \phi(s)$$

[Bertsekas & Ioffe, 1996 (TD); Bertsekas & Tsitsiklis 1996 (TD);Kakade 2002 (policy gradient); Farias & Van Roy, 2006 (approximate LP)]

Function Approximation



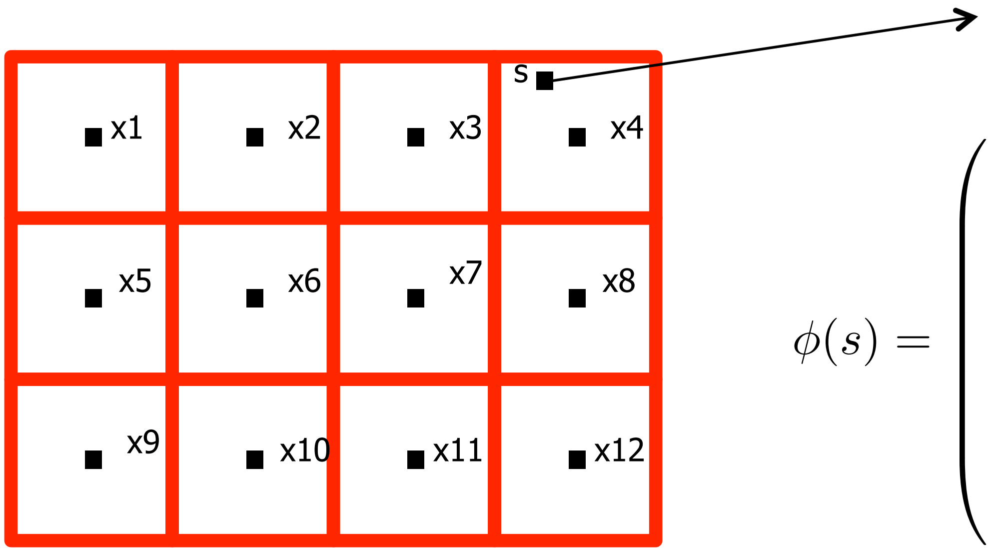
V(s) = θ_0 + θ_1 "distance to closest ghost"
+ θ_2 "distance to closest power pellet"
+ θ_3 "in dead-end"
+ θ_4 "closer to power pellet than ghost is"
+ $\ldots$

= $\sum_{i=0}^{n} \theta_i \phi_i(s) = \theta^\top \phi(s)$

Function Approximation

- 0'th order approximation (1-nearest neighbor):

$$\hat{V}(s) = \hat{V}(x4) = \theta_4$$



$$\phi(s) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix} \qquad \hat{V}(s) = \theta^\top \phi(s)$$

Only store values for x1, x2, …, x12

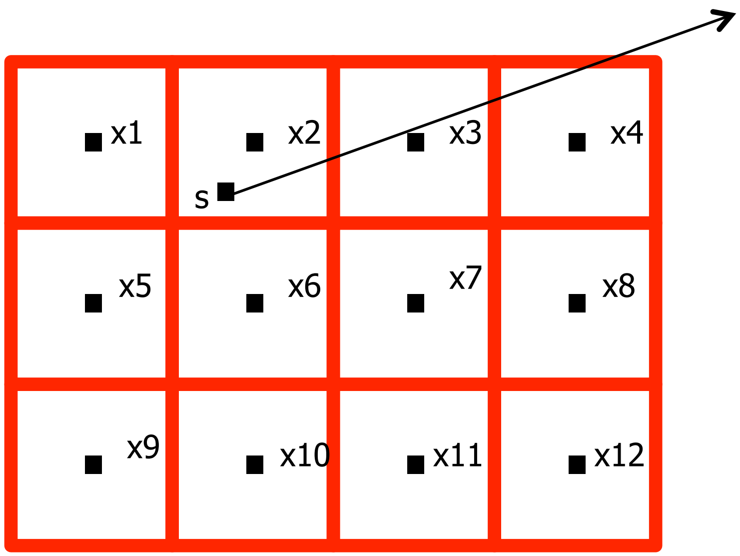
– call these values $\theta_1, \theta_2, \ldots, \theta_{12}$

Assign other states value of nearest "x" state

Function Approximation

- I'th order approximation (k-nearest neighbor interpolation):

$$\hat{V}(s) = \phi_1(s)\theta_1 + \phi_2(s)\theta_2 + \phi_5(s)\theta_5 + \phi_6(s)\theta_6$$



$$\phi(s) = \begin{pmatrix} 0.2 \\ 0.6 \\ 0 \\ 0 \\ 0.05 \\ 0.15 \\ 0 \\ \dots \\ 0 \end{pmatrix} \qquad \hat{V}(s) = \theta^\top \phi(s)$$

Only store values for x1, x2, …, x12

– call these values $\theta_1, \theta_2, \ldots, \theta_{12}$

Assign other states interpolated value of nearest 4 "x" states

Function Approximation

- Examples:
  - $S = \mathbb{R}, \quad \hat{V}(s) = \theta_1 + \theta_2 s$
  - $S = \mathbb{R}, \quad \hat{V}(s) = \theta_1 + \theta_2 s + \theta_3 s^2$
  - $S = \mathbb{R}, \quad \hat{V}(s) = \sum_{i=0}^{n} \theta_i s^i$
  - $S, \quad \hat{V}(s) = \log(\frac{1}{1 + \exp(\theta^\top \phi(s))})$

Function Approximation

- Main idea:
  - Use approximation $\hat{V}_\theta$ of the true value function V,
    - θ is a free parameter to be chosen from its domain Θ
    - Representation size: $|S|$ → downto: $|\Theta|$

    + : less parameters to estimate

    - : less expressiveness, typically there exist many V for which there is no θ such that $\hat{V}_\theta = V$

Supervised Learning

- Given:
  - set of examples

$$(s^{(1)}, V(s^{(1)})), , (s^{(2)}, V(s^{(2)})), \ldots, (s^{(m)}, V(s^{(m)}))$$

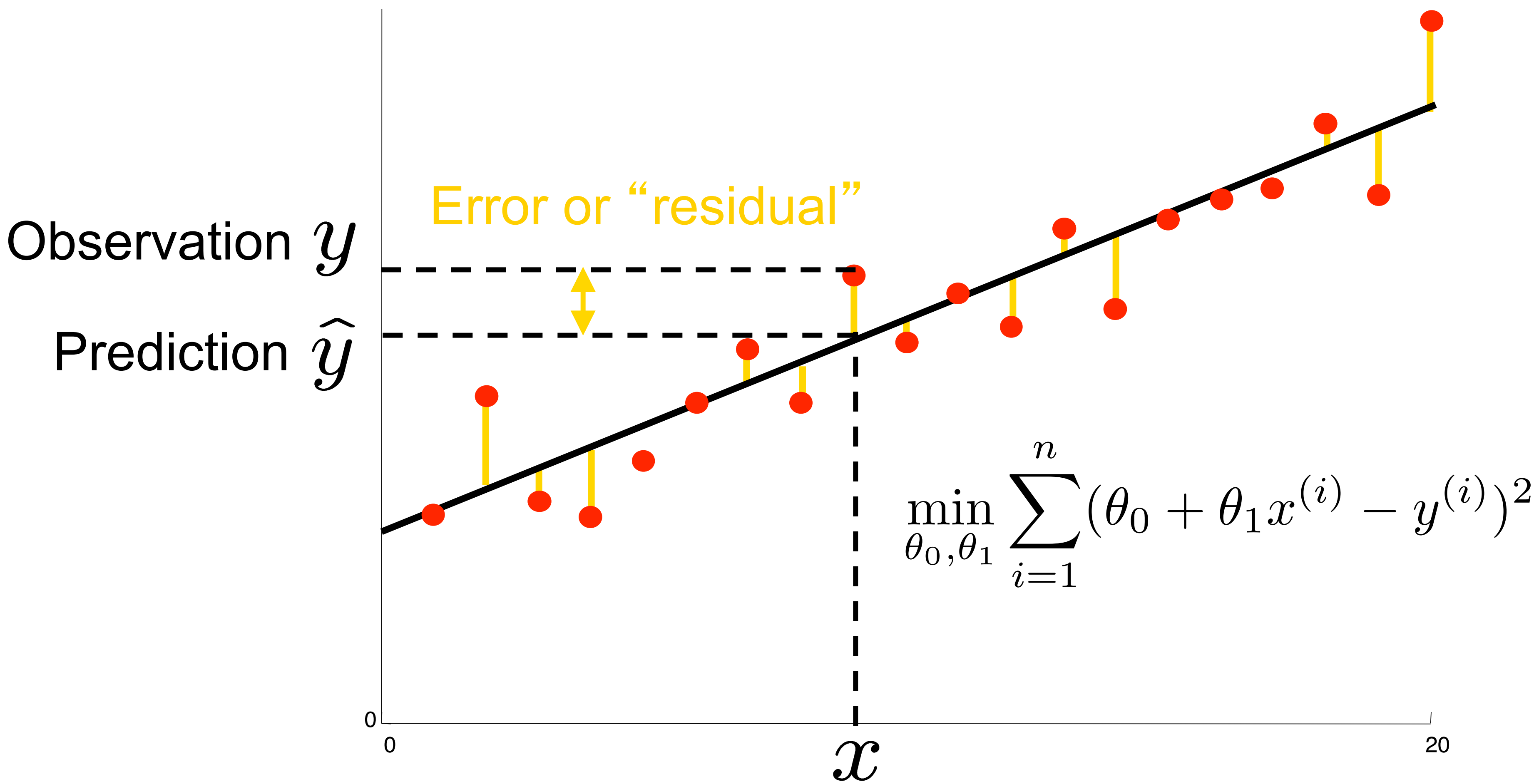
- Asked for:
  - "best" $\hat{V}_\theta$

- Representative approach: find θ through least squares:

$$\min_{\theta \in \Theta} \sum_{i=1}^{m} (\hat{V}_\theta(s^{(i)}) - V(s^{(i)}))^2$$

Supervised Learning Example

- Linear regression



Overfitting

- To avoid overfitting: reduce number of features used

- Practical approach: leave-out validation
  - Perform fitting for different choices of feature sets using just 70% of the data
  - Pick feature set that led to highest quality of fit on the remaining 30% of data

Status

- Function approximation through supervised learning

BUT: where do the supervised examples come from?

Value Iteration with Function Approximation

- Pick some $S' \subseteq S$ (typically $|S'| << |S|$)
- Initialize by choosing some setting for $\theta^{(0)}$
- Iterate for i = 0, 1, 2, …, H:
  - Step 1: Bellman back-ups

$$\forall s \in S' : \quad \bar{V}_{i+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma \hat{V}_{\theta^{(i)}}(s') \right]$$

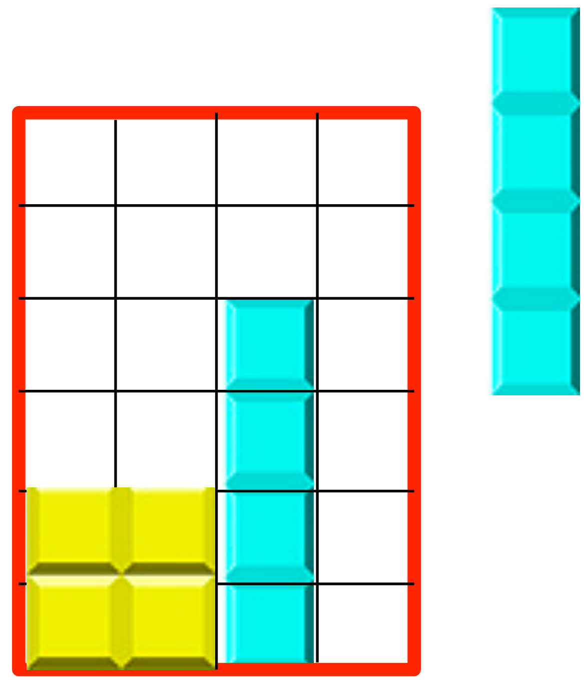
  - Step 2: Supervised learning

    find $\theta^{(i+1)}$ as the solution of:

$$\min_\theta \sum_{s \in S'} \left(\hat{V}_{\theta^{(i+1)}}(s) - \bar{V}_{i+1}(s) \right)^2$$

Value Iteration with Function Approximation --- Example

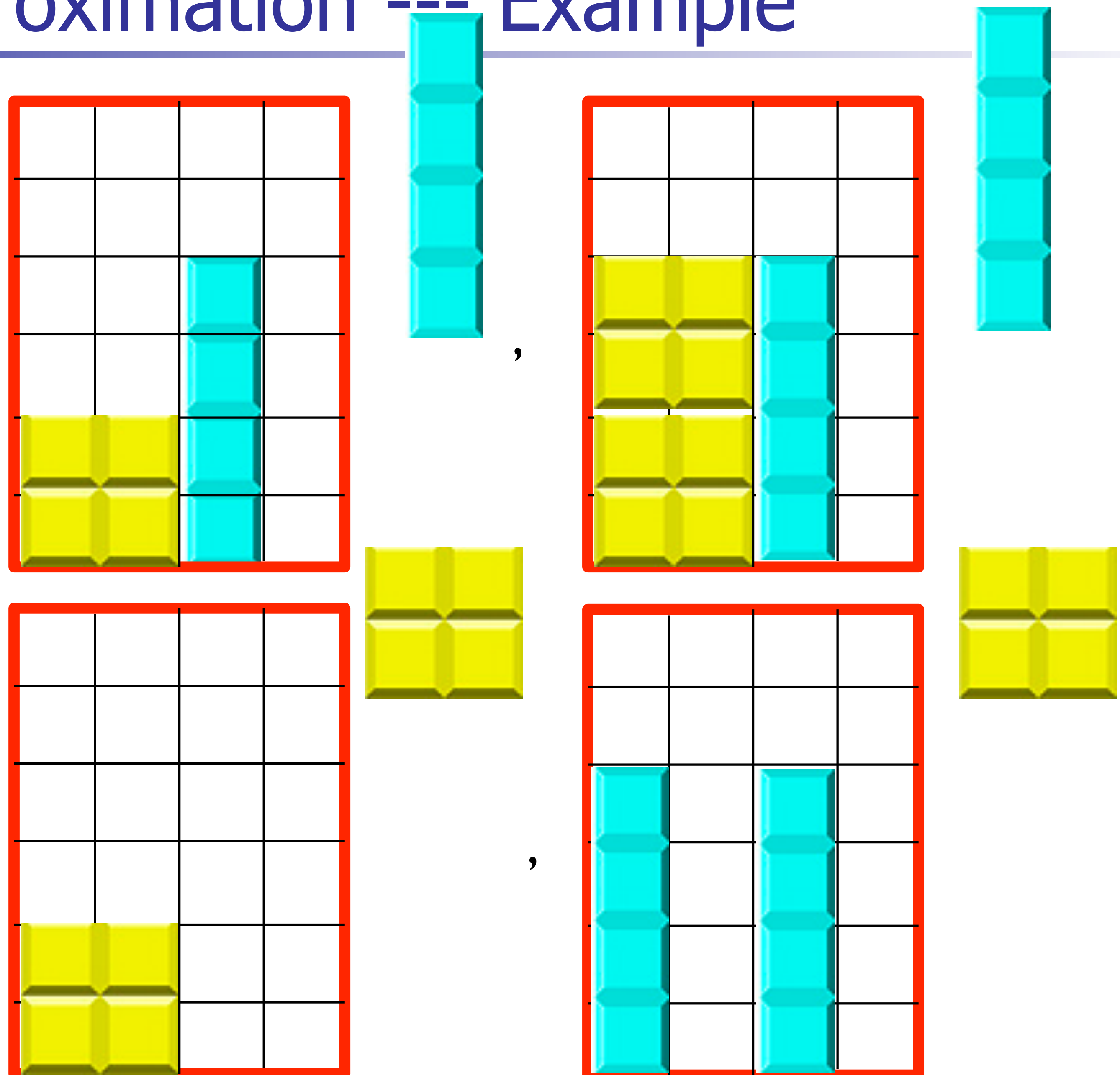
- Mini-tetris: two types of blocks, can only choose translation (not rotation)
  - Example state:



  - Reward = 1 for placing a block
  - Sink state / Game over is reached when block is placed such that part of it extends above the red rectangle
  - If you have a complete row, it gets cleared

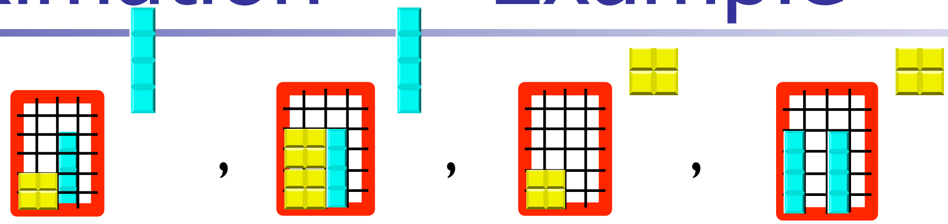
Value Iteration with Function Approximation --- Example

S' = {



}

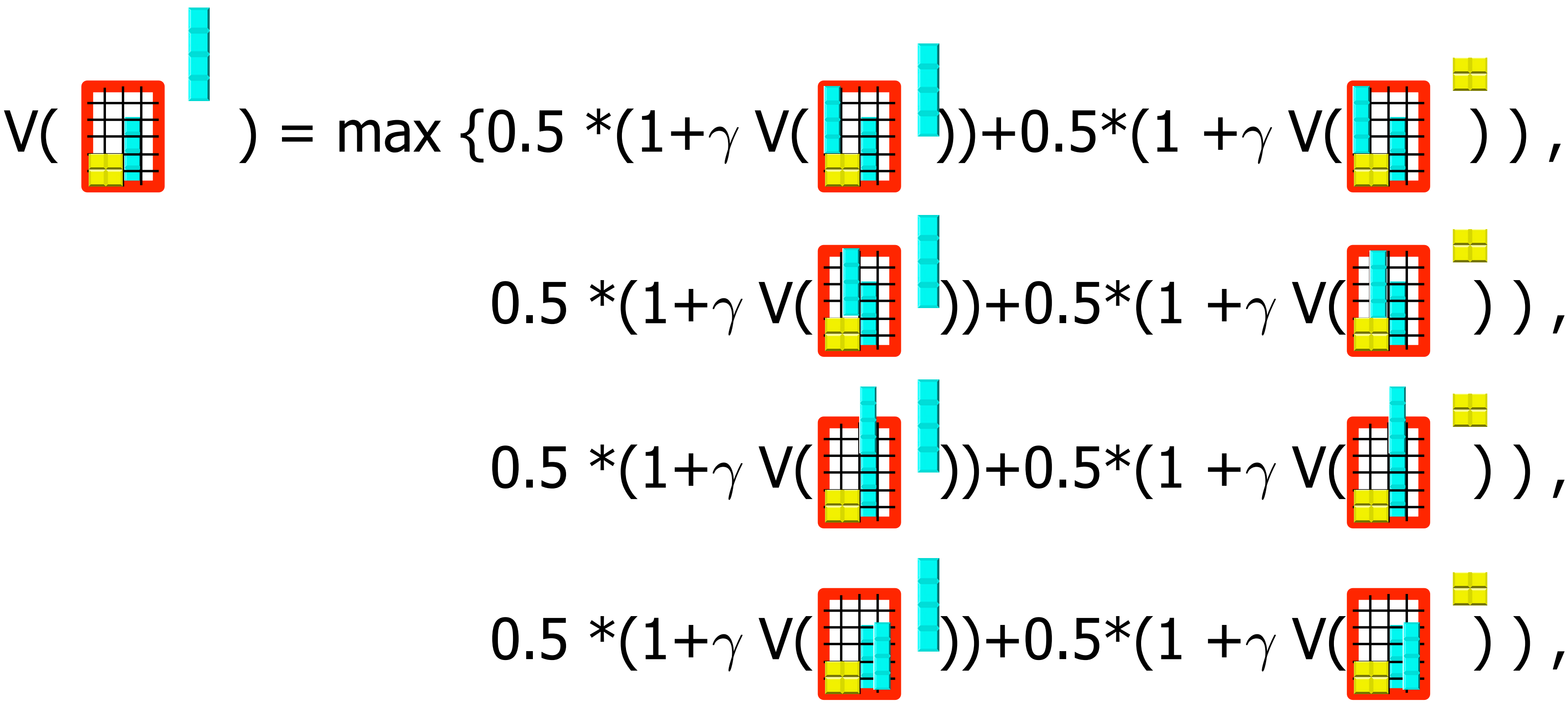
Value Iteration with Function Approximation --- Example

S' = {  }

- 10 features aka basis functions ϕ_i
  - Four basis functions, 0, *. . . , 3, mapping the state to the height h[k] of each of the four* columns.
  - Three basis functions, 4, *. . . , 6, each mapping the state to the absolute difference* between heights of successive columns: *|h[k+1] − h[k]|, k = 1, . . . , 3.*
  - One basis function, 7, that maps state to the maximum column height: $\max_k$ *h[k]*
  - One basis function, 8, that maps state to the number of 'holes' in the board.
  - One basis function, 9, that is equal to 1 in every state.
- Init \theta^{(0)} = (-1, -1, -1, -1, -2, -2, -2, -3, -2, 10)

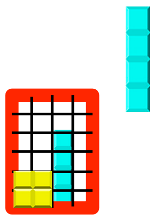
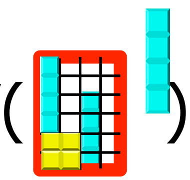
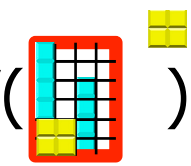
Value Iteration with Function Approximation --- Example

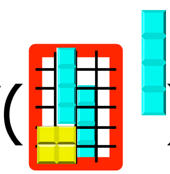
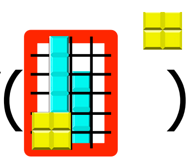
- Bellman back-ups for the states in S':

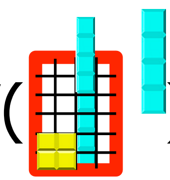
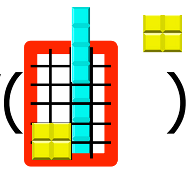


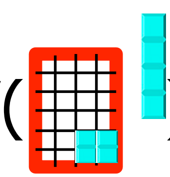
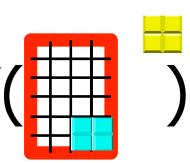
Value Iteration with Function Approximation --- Example

- Bellman back-ups for the states in S':

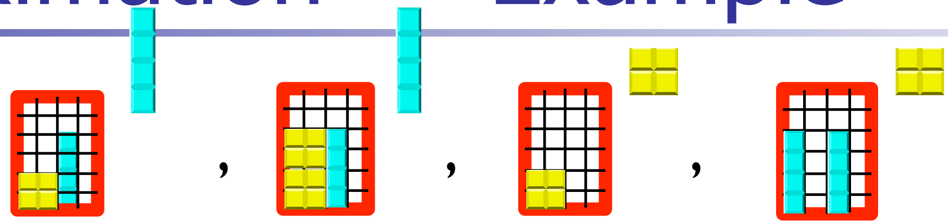
V() = max {0.5 *(1+γ V())+0.5*(1 +γ V()) ,

0.5 *(1+γ V())+0.5*(1 +γ V()) ,

0.5 *(1+γ V())+0.5*(1 +γ V()) ,

0.5 *(1+γ V())+0.5*(1 +γ V()) ,

Value Iteration with Function Approximation --- Example

S' = {  }

- 10 features aka basis functions ϕ_i
  - Four basis functions, 0, . . . , *3, mapping the state to the height h[k] of each of the four* columns.
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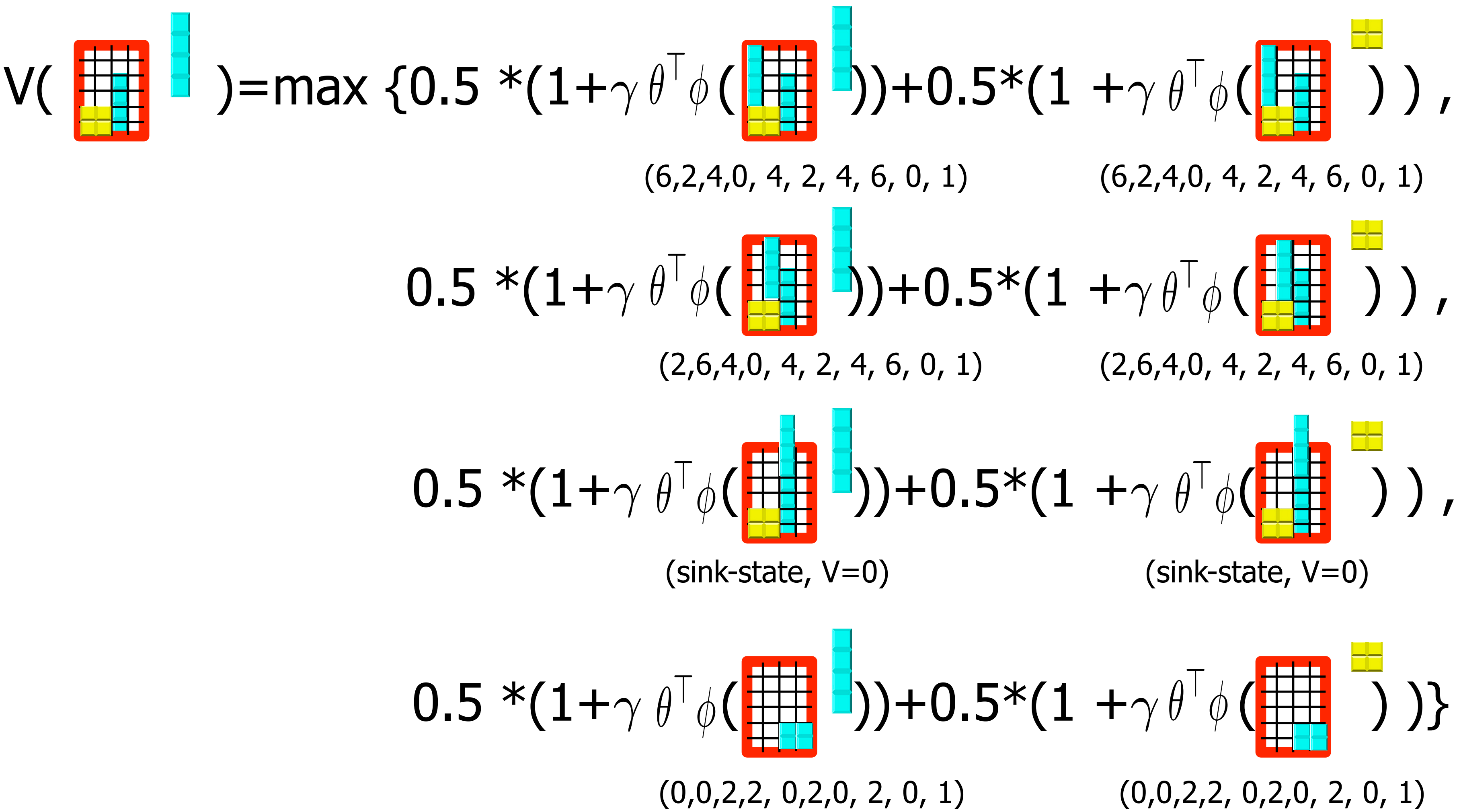
  - One basis function, 7, that maps state to the maximum column height: $\max_k$ *h[k]*
  - One basis function, 8, that maps state to the number of 'holes' in the board.
  - One basis function, 9, that is equal to 1 in every state.
- Init $\theta^{(0)} = (-1, -1, -1, -1, -2, -2, -2, -3, -2, 20)$

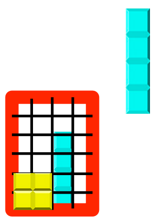
Value Iteration with Function Approximation --- Example

- Bellman back-ups for the states in S':



Value Iteration with Function Approximation --- Example

- Bellman back-ups for the states in S':

V()=max {0.5 *(1+γ -30)+0.5*(1 +γ -30),

0.5 *(1+γ -30)+0.5*(1 +γ -30),

0.5 *(1+γ 0)+0.5*(1 +γ 0),

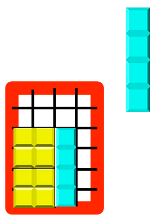
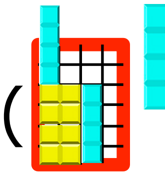
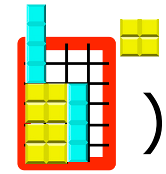
0.5 *(1+γ 6)+0.5*(1 +γ 6),

= 6.4 (for γ = 0.9)

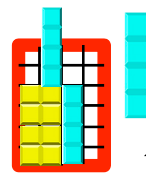
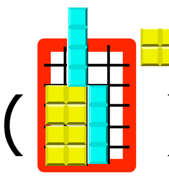
Value Iteration with Function Approximation --- Example

$\theta^{(0)} = (-1, -1, -1, -1, -2, -2, -2, -3, -2, 20)$

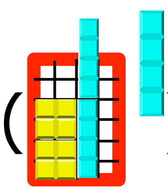
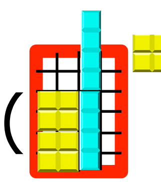
- Bellman back-ups for the second state in S':

V()=max {0.5 *(1+$\gamma\,\theta^\top\phi$())+0.5*(1 +$\gamma\,\theta^\top\phi$()) ,

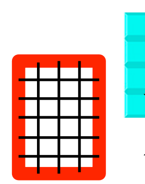
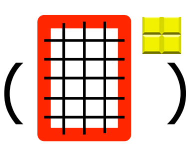
(sink-state, V=0) (sink-state, V=0)

0.5 *(1+$\gamma\,\theta^\top\phi$())+0.5*(1 +$\gamma\,\theta^\top\phi$()) ,

(sink-state, V=0) (sink-state, V=0)

0.5 *(1+$\gamma\,\theta^\top\phi$())+0.5*(1 +$\gamma\,\theta^\top\phi$()) ,

(sink-state, V=0) (sink-state, V=0)

0.5 *(1+$\gamma\,\theta^\top\phi$())+0.5*(1 +$\gamma\,\theta^\top\phi$())}

(0,0,0,0, 0,0,0, 0, 0, 1)
-> V = 20

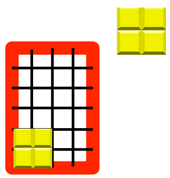
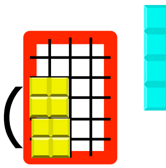
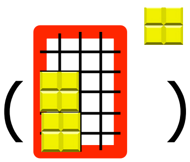
(0,0,0,0, 0,0,0, 0, 0, 1)
-> V = 20

= 19

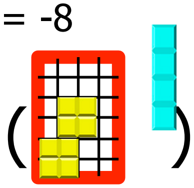
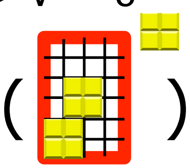
Value Iteration with Function Approximation --- Example

$\theta^{(0)} = (-1,-1,-1,-1,-2,-2,-2,-3,-2,20)$

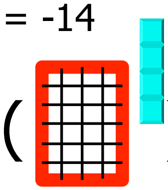
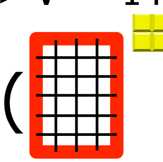
- Bellman back-ups for the third state in S':

V()=max {0.5 *(1+$\gamma\,\theta^\top\phi$())+0.5*(1 +$\gamma\,\theta^\top\phi$()) ,

(4,4,0,0, 0,4,0, 4, 0, 1) -> V = -8 (4,4,0,0, 0,4,0, 4, 0, 1) -> V = -8

0.5 *(1+$\gamma\,\theta^\top\phi$())+0.5*(1 +$\gamma\,\theta^\top\phi$()) ,

(2,4,4,0, 2,0,4, 4, 0, 1) -> V = -14 (2,4,4,0, 2,0,4, 4, 0, 1) -> V = -14

0.5 *(1+$\gamma\,\theta^\top\phi$())+0.5*(1 +$\gamma\,\theta^\top\phi$()) ,

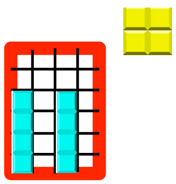
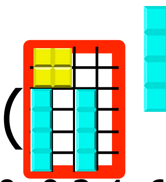
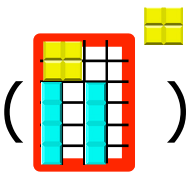
(0,0,0,0, 0,0,0, 0, 0, 1) -> V = 20 (0,0,0,0, 0,0,0, 0, 0, 1) -> V = 20

= 19

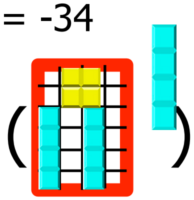
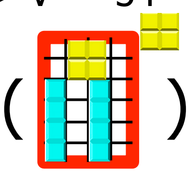
Value Iteration with Function Approximation --- Example

$$\theta^{(0)} = (-1, -1, -1, -1, -2, -2, -2, -3, -2, 20)$$

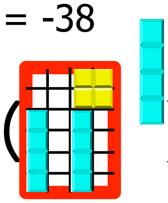
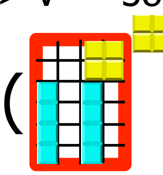
- Bellman back-ups for the fourth state in S':

V()=max {0.5 *(1+$\gamma\, \theta^\top \phi$())+0.5*(1 +$\gamma\, \theta^\top \phi$()) ,

(6,6,4,0, 0,2,4, 6, 4, 1) -> V = -34 | (6,6,4,0, 0,2,4, 6, 4, 1) -> V = -34

0.5 *(1+$\gamma\, \theta^\top \phi$())+0.5*(1 +$\gamma\, \theta^\top \phi$()) ,

(4,6,6,0, 2,0,6, 6, 4, 1) -> V = -38 | (4,6.6,0, 2,0,6, 6, 4, 1) -> V = -38

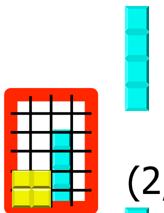
0.5 *(1+$\gamma\, \theta^\top \phi$())+0.5*(1 +$\gamma\, \theta^\top \phi$()) ,

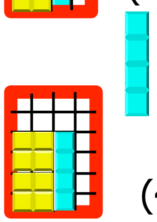
(4,0,6,6, 4,6,0, 6, 4, 1) -> V = -42 | (4,0,6,6, 4,6,0, 6, 4, 1) -> V = -42

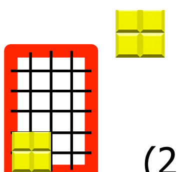
= -29.6

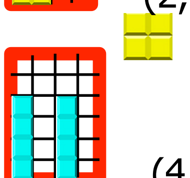
Value Iteration with Function Approximation --- Example

- After running the Bellman back-ups for all 4 states in S' we have:

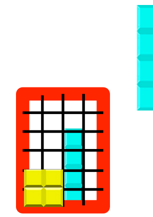
V()= 6.4
(2,2,4,0, 0,2,4, 4, 0, 1)

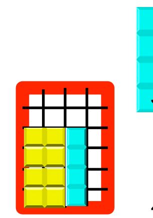
V()= 19
(4,4,4,0, 0,0,4, 4, 0, 1)

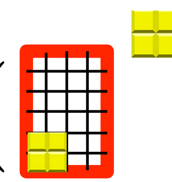
V()= 19
(2,2,0,0, 0,2,0, 2, 0, 1)

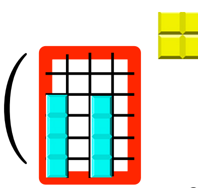
V()= -29.6
(4,0,4,0, 4,4,4, 4, 0, 1)

- We now run supervised learning on these 4 examples to find a new θ:

$\min_\theta (6.4 - \theta^\top \phi($  $))^2$

$+(19 - \theta^\top \phi($  $))^2$

$+(19 - \theta^\top \phi($  $))^2$

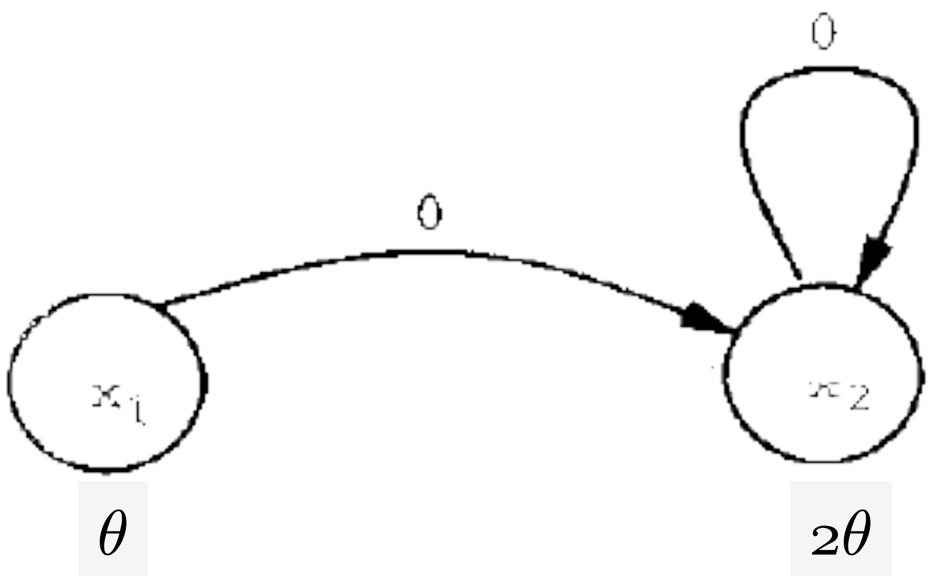
$+((-29.6) - \theta^\top \phi($  $))^2$

→ Running least squares gives new θ

$$\theta^{(1)} = (0.195, 6.24, -2.11, 0, -6.05, 0.13, -2.11, 2.13, 0, 1.59)$$

Potential guarantees?

Simple example**



Function approximator: [1 2] * θ

Simple example**

$$\bar{J}_\theta = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \theta$$

$$\begin{aligned} \bar{J}^{(1)}(x_1) &= 0 + \gamma \hat{J}_{\theta^{(0)}}(x_2) = 2\gamma\theta^{(0)} \\ \bar{J}^{(1)}(x_2) &= 0 + \gamma \hat{J}_{\theta^{(0)}}(x_2) = 2\gamma\theta^{(0)} \end{aligned}$$

Function approximation with least squares fit:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \theta^{(1)} \approx \begin{bmatrix} 2\gamma\theta^{(0)} \\ 2\gamma\theta^{(0)} \end{bmatrix}$$

Least squares fit results in:

$$\theta^{(1)} = \frac{6}{5}\gamma\theta^{(0)}$$

Repeated back-ups and function approximations result in:

$$\theta^{(i)} = \left(\frac{6}{5}\gamma\right)^i \theta^{(0)}$$

which diverges if $\gamma > \frac{5}{6}$ even though the function approximation class can represent the true value function.]

Composing operators**

- **Definition.** An operator G is a *non-expansion* with respect to a norm || . || if

$$\|GJ_1 - GJ_2\| \leq \|J_1 - J_2\|$$

- **Fact.** If the operator F is a γ contraction with respect to a norm || . || and the operator G is a non-expansion with respect to the same norm, then the sequential application of the operators G and F is a γ-contraction, i.e.,

$$\|GFJ_1 - GFJ_2\| \leq \gamma \|J_1 - J_2\|$$

- **Corollary.** If the supervised learning step is a non-expansion, then iteration in value iteration with function approximation is a γ-contraction, and in this case we have a convergence guarantee.

Averager function approximators are non-expansions**

DEFINITION: A real-valued function approximation scheme is an *averager* if every fitted value is the weighted average of zero or more target values and possibly some predetermined constants. The weights involved in calculating the fitted value $\hat{Y}_i$ may depend on the sample vector X_0, but may not depend on the target values Y. More precisely, for a fixed X_0, if Y has n elements, there must exist n real numbers k_i, n^2 nonnegative real numbers β_{ij}, and n nonnegative real numbers β_i, so that for each i we have $\beta_i + \sum_j \beta_{ij} = 1$ and $\hat{Y}_i = \beta_i k_i + \sum_j \beta_{ij} Y_j$.

- Examples:
  - nearest neighbor (aka state aggregation)
  - linear interpolation over triangles (tetrahedrons, …)

Averager function approximators are non-expansions**

Proof: Let J_1 and J_2 be two vectors in $\Re^n$. Consider a particular entry s of ΠJ_1 and ΠJ_2:

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$$
\begin{aligned}
|(\Pi J_1)(s) - (\Pi J_2)(s)| &= |\beta_{s0} + \sum_{s'} \beta_{ss'} J_1(s') - \beta_{s0} + \sum_{s'} \beta_{ss'} J_2(s')| \\
&= |\sum_{s'} \beta_{ss'} (J_1(s') - J_2(s'))| \\
&\leq \max_{s'} |J_1(s') - J_2(s')| \\
&= \|J_1 - J_2\|_\infty
\end{aligned}
$$

This holds true for all s, hence we have

$$\|\Pi J_1 - \Pi J_2\|_\infty \leq \|J_1 - J_2\|_\infty$$

Linear regression ☹ **

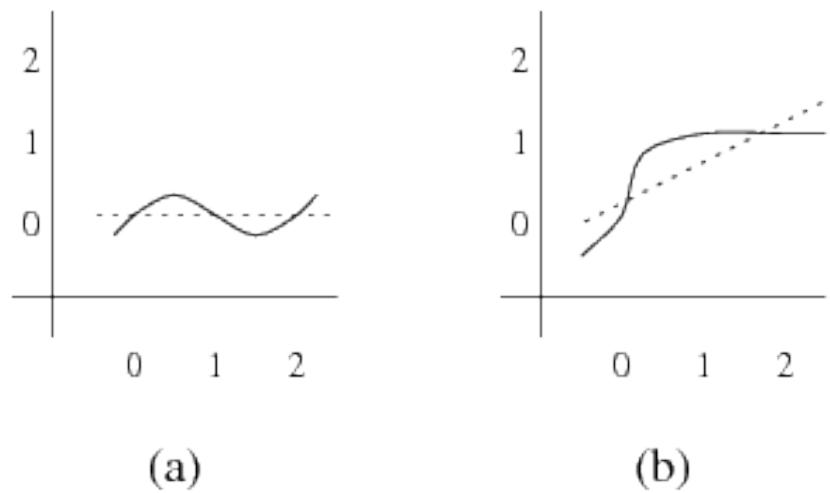

Figure 2: The mapping associated with linear regression when samples are taken at the points $x = 0, 1, 2$. In (a) we see a target value function (solid line) and its corresponding fitted value function (dotted line). In (b) we see another target function and another fitted function. The first target function has values $y = 0, 0, 0$ at the sample points; the second has values $y = 0, 1, 1$. Regression exaggerates the difference between the two functions: the largest difference between the two target functions at a sample point is 1 (at $x = 1$ and $x = 2$), but the largest difference between the two fitted functions at a sample point is $\frac{7}{6}$ (at $x = 2$).

[Example taken from Gordon, 1995.]

Guarantees for fixed point**

Theorem. Let J^* be the optimal value function for a finite MDP with discount factor γ. Let the projection operator Π be a non-expansion w.r.t. the infinity norm and let $\tilde{J}$ be any fixed point of Π. Suppose $\|\tilde{J} - J^*\|_\infty \leq \epsilon$. Then ΠT converges to a value function $\bar{J}$ such that:

$$\|\bar{J} - J^*\| \leq 2\epsilon + \frac{2\gamma\epsilon}{1-\gamma}$$

- I.e., if we pick a non-expansion function approximator which can approximate J* well, then we obtain a good value function estimate.
- To apply to discretization: use continuity assumptions to show that J* can be approximated well by chosen discretization scheme

Outline

- Value iteration with function approximation
- Linear programming with function approximation

Infinite Horizon Linear Program

$$\min_{V} \sum_{s \in S} \mu_0(s) V(s)$$

$$\text{s.t.} \quad V(s) \geq \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V(s') \right], \quad \forall s \in S, a \in A$$

μ_0 is a probability distribution over S, with μ_0(s)> 0 for all s $\in$ S.

Theorem. V* is the solution to the above LP.

Infinite Horizon Linear Program

$$\min_V \sum_{s \in S} \mu_0(s) V(s)$$

$$\text{s.t.} \quad V(s) \geq \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V(s') \right], \quad \forall s \in S, a \in A$$

- Let: $V(s) = \theta^\top \phi(s)$, and consider S' rather than S:

$$\min_\theta \sum_{s \in S'} \mu_0(s) \theta^\top \phi(s)$$

$$\text{s.t.} \quad \theta^\top \phi(s) \geq \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma \theta^\top \phi(s') \right], \quad \forall s \in S', a \in A$$

→ Linear program that finds $\hat{V}_\theta(s) = \theta^\top \phi(s)$

Approximate Linear Program – Guarantees**

$$\min_{\theta} \sum_{s \in S'} \mu_0(s) \theta^\top \phi(s)$$

$$\text{s.t.} \quad \theta^\top \phi(s) \geq \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma \theta^\top \phi(s') \right], \quad \forall s \in S', a \in A$$

- LP solver will converge
- Solution quality: [de Farias and Van Roy, 2002]

  Assuming one of the features is the feature that is equal to one for all states, and assuming S'=S we have that:

$$\|V^* - \Phi\theta\|_{1,\mu_0} \leq \frac{2}{1-\gamma} \min_{\theta} \|V^* - \Phi\theta\|_\infty$$

  (slightly weaker, probabilistic guarantees hold for S' not equal to S, these guarantees require size of S' to grow as the number of features grows)