Gaussians

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Many slides adapted from Thrun, Burgard and Fox, Probabilistic Robotics

Outline

- Univariate Gaussian
- Multivariate Gaussian
- Law of Total Probability
- Conditioning (Bayes' rule)

Disclaimer: lots of linear algebra in next few lectures. See course homepage for pointers for brushing up your linear algebra.

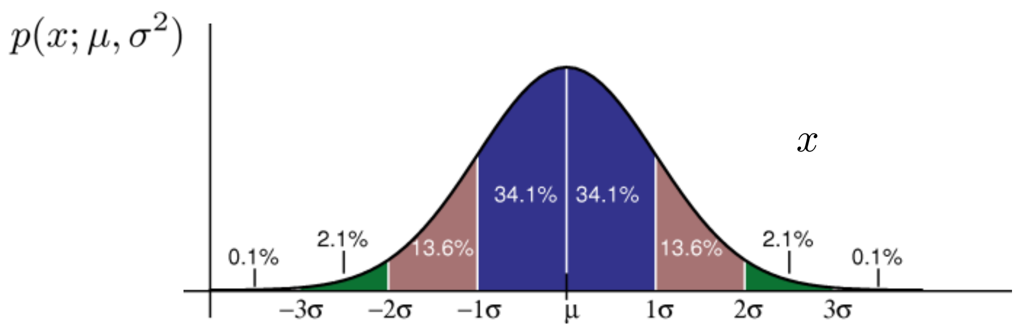
In fact, pretty much all computations with Gaussians will be reduced to linear algebra!

Univariate Gaussian

- Gaussian distribution with mean μ, and standard deviation σ:

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

$$p(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$



$X \sim \mathcal{N}(\mu, \sigma^2)$

Properties of Gaussians

$p(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$

- Densities integrate to one: $\int_{-\infty}^{\infty} p(x; \mu, \sigma^2)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})dx = 1$

- Mean:
$$\begin{aligned} \mathsf{E}_X[X] &= \int_{-\infty}^{\infty} x p(x; \mu, \sigma^2)dx \\ &= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})dx \\ &= \mu \end{aligned}$$

- Variance:
$$\begin{aligned} \mathsf{E}_X[(X-\mu)^2] &= \int_{-\infty}^{\infty} (x-\mu)^2 p(x; \mu, \sigma^2)dx \\ &= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})dx \\ &= \sigma^2 \end{aligned}$$

Central limit theorem (CLT)

- Classical CLT:
  - Let $X_1, X_2, \ldots$ be an infinite sequence of *independent* random variables with $E\,X_i = \mu$, $E(X_i - \mu)^2 = \sigma^2$
  - Define $Z_n = ((X_1 + \ldots + X_n) - n\,\mu) / (\sigma\, n^{1/2})$
  - Then for the limit of n going to infinity we have that Z_n is distributed according to N(0,1)
- Crude statement: things that are the result of the addition of lots of small effects tend to become Gaussian.

Multivariate Gaussians

$$p(x; \mu, \Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right)$$

$$\int \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right) dx = 1$$

For a matrix $A \in \mathbb{R}^{n\times n}$, $|A|$ denotes the determinant of A.

For a matrix $A \in \mathbb{R}^{n\times n}$, A^{-1} denotes the inverse of A, which satisfies $A^{-1}A = I = AA^{-1}$ with $I \in \mathbb{R}^{n\times n}$ the identity matrix with all diagonal entries equal to one, and all off-diagonal entries equal to zero.

Hint: often when trying to understand matrix equations, it's easier to first consider the special case of the dimensions of the matrices being one-by-one. Once parsing them that way makes sense, a good second step can be to parse them assuming all matrices are diagonal matrices. Once parsing them that way makes sense, usually it is only a small step to understand the general case.

Multivariate Gaussians

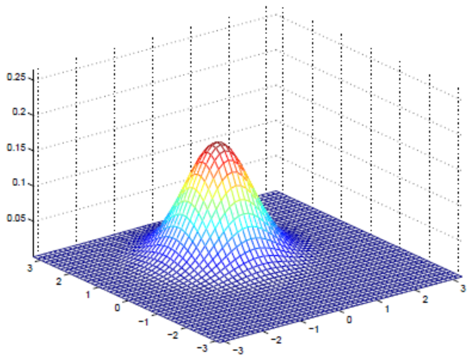
$$
\begin{aligned}
\mathsf{E}_X[X_i] &= \int x_i p(x;\mu,\Sigma)dx = \mu_i \\
\mathsf{E}_X[X] &= \int x p(x;\mu,\Sigma)dx = \mu
\end{aligned}
$$

(integral of vector = vector of integrals of each entry)

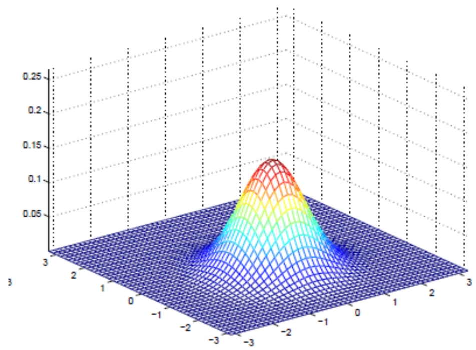
$$
\begin{aligned}
\mathsf{E}_X[(X_i-\mu_i)(X_j-\mu_j)] &= \int (x_i-\mu_i)(x_j-\mu_j)p(x;\mu,\Sigma)dx = \Sigma_{ij} \\
\mathsf{E}_X[(X-\mu)(X-\mu)^\top] &= \int [(X-\mu)(X-\mu)^\top p(x;\mu,\Sigma)dx = \Sigma
\end{aligned}
$$

(integral of matrix = matrix of integrals of each entry)

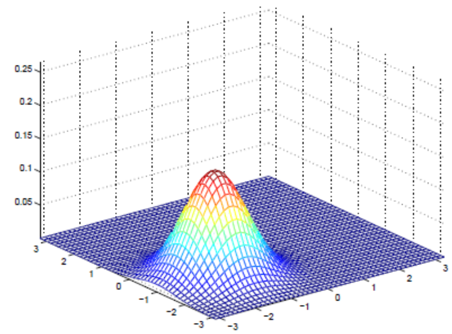
Multivariate Gaussians: Examples



- μ = [1; 0]
- Σ = [1 0; 0 1]

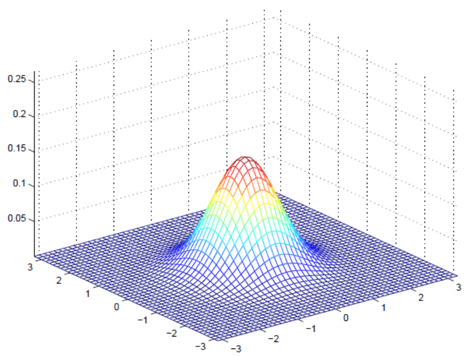


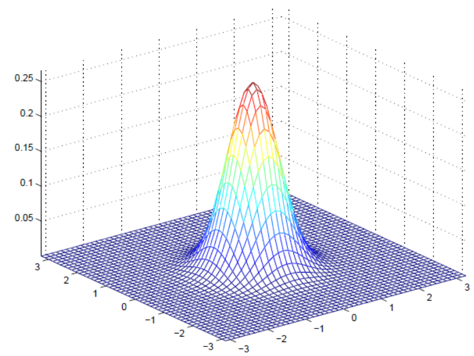
- μ = [-.5; 0]
- Σ = [1 0; 0 1]

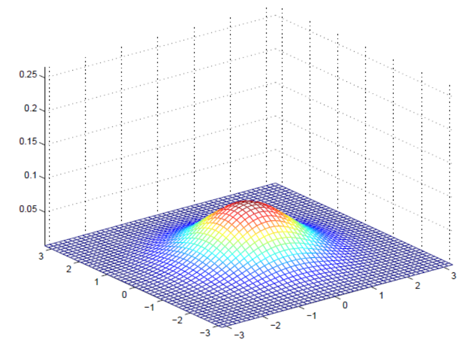


- μ = [-1; -1.5]
- Σ = [1 0; 0 1]

Multivariate Gaussians: Examples





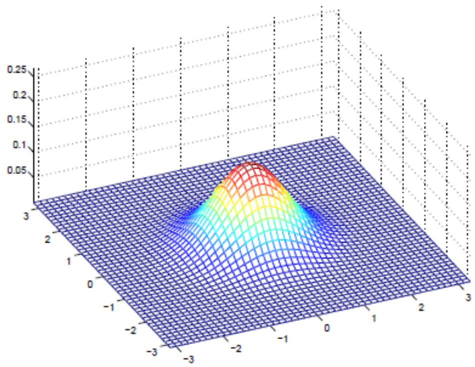


- μ = [0; 0]
- Σ = [1 0 ; 0 1]

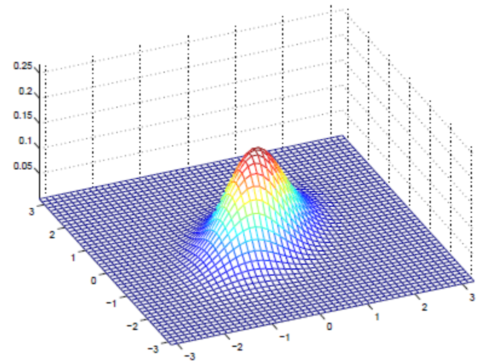
- μ = [0; 0]
- Σ = [.6 0 ; 0 .6]

- μ = [0; 0]
- Σ = [2 0 ; 0 2]

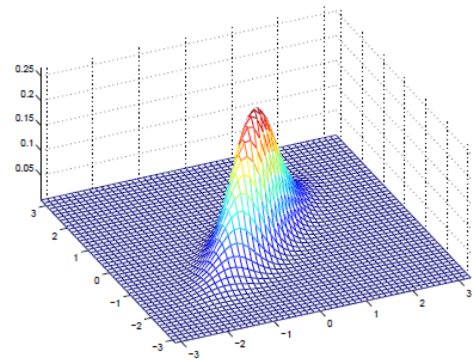
Multivariate Gaussians: Examples



- μ = [0; 0]
- Σ = [1 0; 0 1]

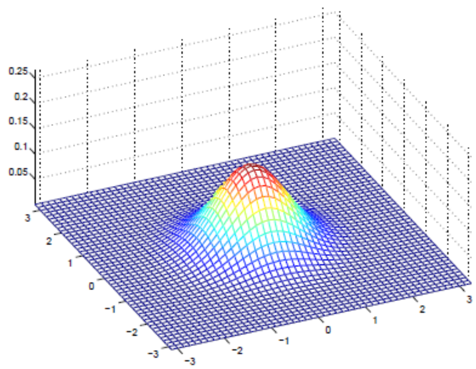


- μ = [0; 0]
- Σ = [1 0.5; 0.5 1]

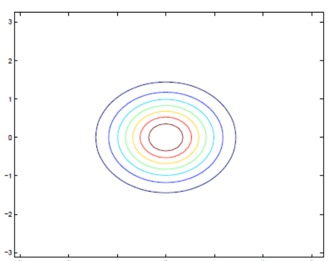


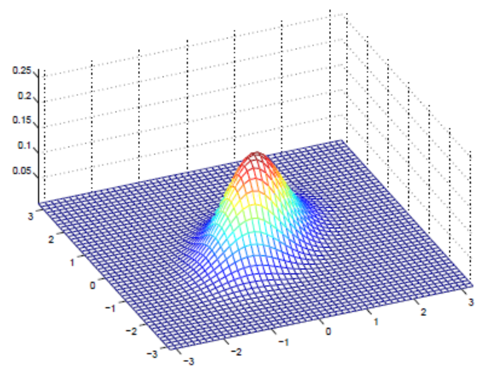
- μ = [0; 0]
- Σ = [1 0.8; 0.8 1]

Multivariate Gaussians: Examples

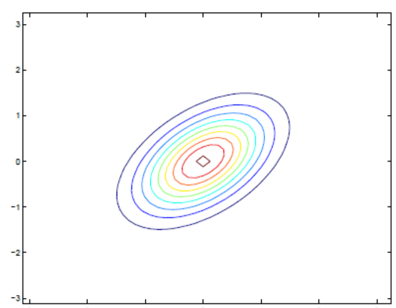


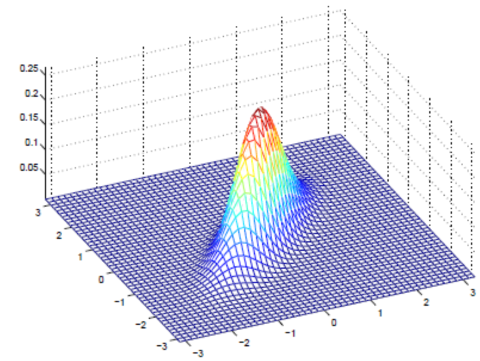
- $\mu = [0; 0]$
- $\Sigma = [1 \;\; 0; 0 \;\; 1]$



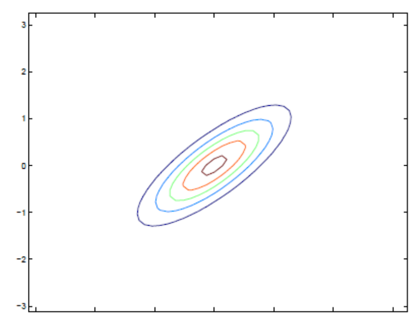


- $\mu = [0; 0]$
- $\Sigma = [1 \;\; 0.5; 0.5 \;\; 1]$

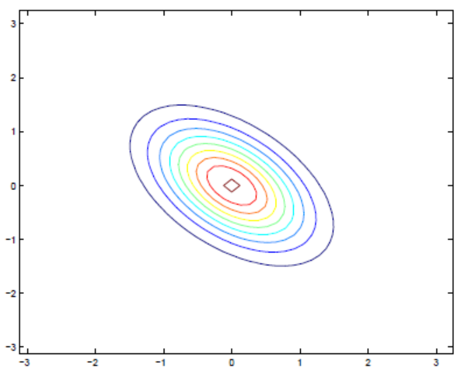




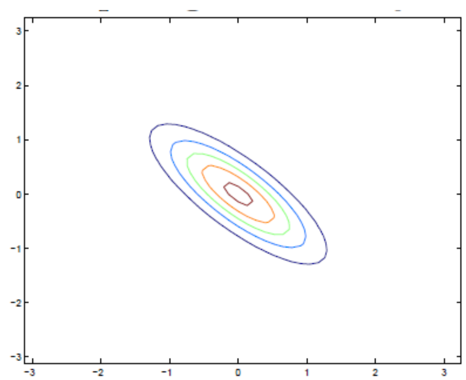
- $\mu = [0; 0]$
- $\Sigma = [1 \;\; 0.8; 0.8 \;\; 1]$



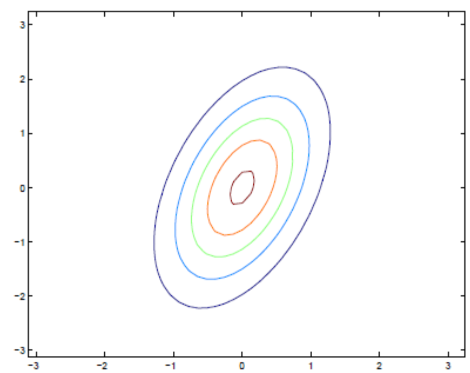
Multivariate Gaussians: Examples



- μ = [0; 0]
- Σ = [1 -0.5 ; -0.5 1]



- μ = [0; 0]
- Σ = [1 -0.8 ; -0.8 1]



- μ = [0; 0]
- Σ = [3 0.8 ; 0.8 1]

Partitioned Multivariate Gaussian

- Consider a multi-variate Gaussian and partition random vector into (X, Y).

$$\mathcal{N}(\mu, \Sigma) = \mathcal{N}\left(\begin{bmatrix}\mu_X \\ \mu_Y\end{bmatrix}, \begin{bmatrix}\Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY}\end{bmatrix}\right)$$

$$p(\begin{bmatrix}x \\ y\end{bmatrix}; \mu, \Sigma) = \frac{1}{(2\pi)^(n/2)|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}\left(\begin{bmatrix}x \\ y\end{bmatrix} - \begin{bmatrix}\mu_X \\ \mu_Y\end{bmatrix}\right)^\top \begin{bmatrix}\Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY}\end{bmatrix}^{-1} \left(\begin{bmatrix}x \\ y\end{bmatrix} - \begin{bmatrix}\mu_X \\ \mu_Y\end{bmatrix}\right)\right)$$

$$\begin{aligned}
\mu_X &= \mathrm{E}_{(X,Y)\sim\mathcal{N}(\mu,\Sigma)}[X] \\
\mu_Y &= \mathrm{E}_{(X,Y)\sim\mathcal{N}(\mu,\Sigma)}[Y] \\
\Sigma_{XX} &= \mathrm{E}_{(X,Y)\sim\mathcal{N}(\mu,\Sigma)}[(X-\mu_X)(X-\mu_X)^\top] \\
\Sigma_{YY} &= \mathrm{E}_{(X,Y)\sim\mathcal{N}(\mu,\Sigma)}[(Y-\mu_Y)(Y-\mu_Y)^\top] \\
\Sigma_{XY} &= \mathrm{E}_{(X,Y)\sim\mathcal{N}(\mu,\Sigma)}[(X-\mu_X)(Y-\mu_Y)^\top] = \Sigma_{YX}^\top \\
\Sigma_{YX} &= \mathrm{E}_{(X,Y)\sim\mathcal{N}(\mu,\Sigma)}[(Y-\mu_Y)(X-\mu_X)^\top] = \Sigma_{XY}^\top
\end{aligned}$$

Partitioned Multivariate Gaussian: Dual Representation

- Precision matrix $\Gamma = \Sigma^{-1} = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix}^{-1} = \begin{bmatrix} \Gamma_{XX} & \Gamma_{XY} \\ \Gamma_{YX} & \Gamma_{YY} \end{bmatrix}$ (1)

$$p(\begin{bmatrix} x \\ y \end{bmatrix}; \mu, \Sigma) = \frac{1}{(2\pi)^(n/2)|\Sigma|^{1/2}} \exp\left(-\frac{1}{2} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \right)^\top \begin{bmatrix} \Gamma_{XX} & \Gamma_{XY} \\ \Gamma_{YX} & \Gamma_{YY} \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \right) \right)$$

- Straightforward to verify from (1) that:

$$\begin{aligned}
\Sigma_{XX} &= \left(\Gamma_{XX} - \Gamma_{XY}\Gamma_{YY}^{-1}\Gamma_{YX}\right)^{-1} \\
\Sigma_{YY} &= \left(\Gamma_{YY} - \Gamma_{YX}\Gamma_{XX}^{-1}\Gamma_{XY}\right)^{-1} \\
\Sigma_{XY} &= -\Gamma_{XX}^{-1}\Gamma_{XY}\left(\Gamma_{YY} - \Gamma_{YX}\Gamma_{XX}^{-1}\Gamma_{XY}\right)^{-1} = \Sigma_{YX}^\top \\
\Sigma_{YX} &= -\Gamma_{YY}^{-1}\Gamma_{YX}\left(\Gamma_{XX} - \Gamma_{XY}\Gamma_{YY}^{-1}\Gamma_{YX}\right)^{-1} = \Sigma_{XY}^\top
\end{aligned}$$

- And swapping the roles of Sigma and Gamma:

$$\begin{aligned}
\Gamma_{XX} &= \left(\Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}\right)^{-1} \\
\Gamma_{YY} &= \left(\Sigma_{YY} - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}\right)^{-1} \\
\Gamma_{XY} &= -\Sigma_{XX}^{-1}\Sigma_{XY}\left(\Sigma_{YY} - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}\right)^{-1} = \Gamma_{YX}^\top \\
\Gamma_{YX} &= -\Sigma_{YY}^{-1}\Sigma_{YX}\left(\Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}\right)^{-1} = \Gamma_{XY}^\top
\end{aligned}$$

Marginalization: p(x) = ?

$$p(\begin{bmatrix} x \\ y \end{bmatrix}; \mu, \Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}\right)^\top \begin{bmatrix} \Gamma_{XX} & \Gamma_{XY} \\ \Gamma_{YX} & \Gamma_{YY} \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}\right)\right)$$

We integrate out over y to find the marginal:

$$\begin{aligned}
p(x) &= \int p(\begin{bmatrix} x \\ y \end{bmatrix}; \mu, \Sigma) dy \\
&= \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \int \exp\left(-\frac{1}{2}\left((x-\mu_X)^\top \Gamma_{XX}(x-\mu_X) + (y-\mu_Y)^\top \Gamma_{YY}(y-\mu_Y) + 2(y-\mu_Y)^\top \Gamma_{YX}(x-\mu_X)\right)\right) dy \\
&= \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \int \exp\left(-\frac{1}{2}\left((x-\mu_X)^\top \Gamma_{XX}(x-\mu_X) + (y-\mu_Y)^\top \Gamma_{YY}(y-\mu_Y) + 2(y-\mu_Y)^\top \Gamma_{YY}\Gamma_{YY}^{-1}\Gamma_{YX}(x-\mu_X) + (x-\mu_X)^\top \Gamma_{XY}\Gamma_{YY}^{-1}\Gamma_{YY}\Gamma_{YY}^{-1}\Gamma_{YX}(x-\mu_X) - (x-\mu_X)^\top \Gamma_{XY}\Gamma_{YY}^{-1}\Gamma_{YY}\Gamma_{YY}^{-1}\Gamma_{YX}(x-\mu_X)\right)\right) dy \\
&= \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}\left((x-\mu_X)^\top \Gamma_{XX}(x-\mu_X) - (x-\mu_X)^\top \Gamma_{XY}\Gamma_{YY}^{-1}\Gamma_{YY}\Gamma_{YY}^{-1}\Gamma_{YX}(x-\mu_X)\right)\right) \int \exp\left(-\frac{1}{2}\left((y-\mu_Y)^\top \Gamma_{YY}(y-\mu_Y) + 2(y-\mu_Y)^\top \Gamma_{YY}\Gamma_{YY}^{-1}\Gamma_{YX}(x-\mu_X) + (x-\mu_X)^\top \Gamma_{XY}\Gamma_{YY}^{-1}\Gamma_{YY}\Gamma_{YY}^{-1}\Gamma_{YX}(x-\mu_X)\right)\right) dy \\
&= \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}\left((x-\mu_X)^\top \Gamma_{XX}(x-\mu_X) - (x-\mu_X)^\top \Gamma_{XY}\Gamma_{YY}^{-1}\Gamma_{YX}(x-\mu_X)\right)\right) \int \exp\left(-\frac{1}{2}\left((y-\mu_Y + \Gamma_{YY}^{-1}\Gamma_{YX}(x-\mu_X))^\top \Gamma_{YY}(y-\mu_Y + \Gamma_{YY}^{-1}\Gamma_{YX}(x-\mu_X))\right)\right) dy \\
&= \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}\left((x-\mu_X)^\top \Gamma_{XX}(x-\mu_X) - (x-\mu_X)^\top \Gamma_{XY}\Gamma_{YY}^{-1}\Gamma_{YX}(x-\mu_X)\right)\right) (2\pi)^{n_Y/2}|\Gamma_{YY}^{-1}|^{1/2} \\
&= \frac{(2\pi)^{n_Y/2}|\Gamma_{YY}^{-1}|^{1/2}}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}\left((x-\mu_X)^\top \Gamma_{XX}(x-\mu_X) - (x-\mu_X)^\top \Gamma_{XY}\Gamma_{YY}^{-1}\Gamma_{YX}(x-\mu_X)\right)\right) \\
&= \frac{(2\pi)^{n_Y/2}|\Gamma_{YY}^{-1}|^{1/2}}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}\left((x-\mu_X)^\top \left(\Gamma_{XX} - \Gamma_{XY}\Gamma_{YY}^{-1}\Gamma_{YX}\right)(x-\mu_X)\right)\right)
\end{aligned}$$

Hence we have:

$$X \sim \mathcal{N}(\mu_X, (\Gamma_{XX} - \Gamma_{XY}\Gamma_{YY}^{-1}\Gamma_{YX})^{-1}) = \mathcal{N}(\mu_X, \Sigma_{XX})$$

Note: ***if*** *we had known beforehand* that p(x) would be a Gaussian distribution, then we could have found the result more quickly. We would have just needed to find $\mu_X = \mathrm{E}[X]$ and $\Sigma_{XX} = \mathrm{E}[(X-\mu_X)(X-\mu_X)^\top]$, which we had available through $\mathcal{N}(\mu, \Sigma)$

Marginalization Recap

If

$$(X, Y) \sim \mathcal{N}(\mu, \Sigma) = \mathcal{N}\left(\begin{bmatrix}\mu_X \\ \mu_Y\end{bmatrix}, \begin{bmatrix}\Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY}\end{bmatrix}\right)$$

Then

$$\begin{aligned} X &\sim \mathcal{N}(\mu_X, \Sigma_{XX}) \\ Y &\sim \mathcal{N}(\mu_Y, \Sigma_{YY}) \end{aligned}$$

Self-quiz

Test your understanding of the completion of squares trick! Let $A \in \mathbf{R}^{n\times n}$ be a positive definite matrix, $b \in \mathbf{R}^n$, and $c \in \mathbf{R}$. Prove that

$$\int_{x\in\mathbf{R}^n} \exp\left(-\frac{1}{2}x^T Ax - x^T b - c\right)dx$$

$$= \frac{(2\pi)^{n/2}}{|A|^{1/2}\exp(c - \frac{1}{2}b^T A^{-1}b)}.$$

Conditioning: $p(x \mid Y = y_0) = ?$

$$p\left(\begin{bmatrix} x \\ y \end{bmatrix}; \mu, \Sigma\right) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}\right)^\top \begin{bmatrix} \Gamma_{XX} & \Gamma_{XY} \\ \Gamma_{YX} & \Gamma_{YY} \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}\right)\right)$$

We have

$$\begin{aligned}
p(x|Y=y_0) &\propto p\left(\begin{bmatrix} x \\ y_0 \end{bmatrix}; \mu, \Sigma\right) \\
&\propto \exp\left(-\frac{1}{2}(x-\mu_X)^\top\Gamma_{XX}(x-\mu_X) - (x-\mu_X)^\top\Gamma_{XY}(y_0-\mu_Y) - \frac{1}{2}(y_0-\mu_Y)^\top\Gamma_{YY}(y_0-\mu_Y)\right) \\
&\propto \exp\left(-\frac{1}{2}(x-\mu_X)^\top\Gamma_{XX}(x-\mu_X) - (x-\mu_X)^\top\Gamma_{XY}(y_0-\mu_Y)\right) \\
&= \exp\left(-\frac{1}{2}(x-\mu_X)^\top\Gamma_{XX}(x-\mu_X) - (x-\mu_X)^\top\Gamma_{XX}\Gamma_{XX}^{-1}\Gamma_{XY}(y_0-\mu_Y) - \frac{1}{2}(y_0-\mu_Y)\Gamma_{YX}\Gamma_{XX}^{-1}\Gamma_{XX}\Gamma_{XX}^{-1}\Gamma_{XY}(y_0-\mu_Y) + \frac{1}{2}(y_0-\mu_Y)\Gamma_{YX}\Gamma_{XX}^{-1}\Gamma_{XX}\Gamma_{XX}^{-1}\Gamma_{XY}(y_0-\mu_Y)\right) \\
&= \exp\left(-\frac{1}{2}(x-\mu_X+\Gamma_{XX}^{-1}\Gamma_{XY}(y_0-\mu_Y)^\top\Gamma_{XX}(x-\mu_X+\Gamma_{XX}^{-1}\Gamma_{XY}(y_0-\mu_Y)\right)\exp\left(\frac{1}{2}(y_0-\mu_Y)\Gamma_{YX}\Gamma_{XX}^{-1}\Gamma_{XX}\Gamma_{XX}^{-1}\Gamma_{XY}(y_0-\mu_Y)\right) \\
&\propto \exp\left(-\frac{1}{2}(x-\mu_X+\Gamma_{XX}^{-1}\Gamma_{XY}(y_0-\mu_Y)^\top\Gamma_{XX}(x-\mu_X+\Gamma_{XX}^{-1}\Gamma_{XY}(y_0-\mu_Y)\right)
\end{aligned}$$

Hence we have:

$$\begin{aligned}
X|Y=y_0 &\sim \mathcal{N}(\mu_X - \Gamma_{XX}^{-1}\Gamma_{XY}(y_0-\mu_Y), \Gamma_{XX}^{-1}) \\
&= \mathcal{N}(\mu_X + \Sigma_{XY}\Sigma_{YY}^{-1}(y_0-\mu_Y), \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX})
\end{aligned}$$

- Conditional mean moved according to correlation and variance on measurement
- Conditional covariance does not depend on y_0

Conditioning Recap

If

$$(X, Y) \sim \mathcal{N}(\mu, \Sigma) = \mathcal{N}\left(\begin{bmatrix}\mu_X \\ \mu_Y\end{bmatrix}, \begin{bmatrix}\Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY}\end{bmatrix}\right)$$

Then

$$\begin{aligned} X|Y = y_0 &\sim \mathcal{N}(\mu_X + \Sigma_{XY}\Sigma_{YY}^{-1}(y_0 - \mu_Y), \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}) \\ Y|X = x_0 &\sim \mathcal{N}(\mu_Y + \Sigma_{YX}\Sigma_{XX}^{-1}(x_0 - \mu_X), \Sigma_{YY} - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}) \end{aligned}$$