Kalman Filtering

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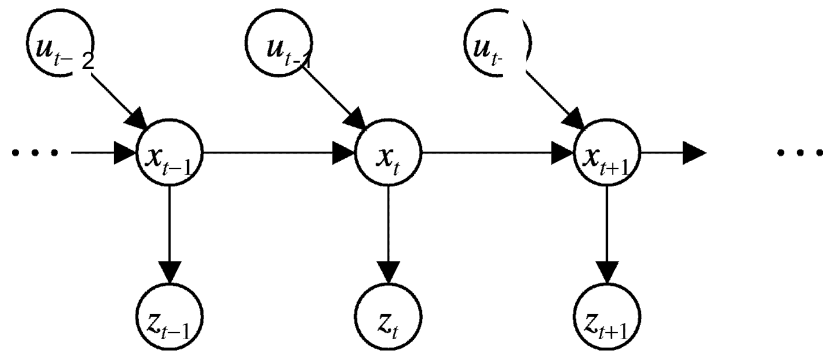
Many slides adapted from Thrun, Burgard and Fox, Probabilistic Robotics

Overview

- Kalman Filter = special case of a Bayes' filter with dynamics model and sensory model being linear Gaussian:

$$X_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$$



$$p(x_t|x_{0:t-1}, z_{0:t-1}, u_{0:t-1}) = p(x_t|x_{t-1}, u_{t-1}) \sim \mathcal{N}(A_{t-1}x_{t-1} + B_{t-1}u_{t-1}, Q_{t-1})$$
$$p(z_t|x_{0:t}, z_{0:t-1}, u_{0:t-1}) = p(z_t|x_t) \sim \mathcal{N}(C_t x_t + d_t, R_t)$$

- Above can also be written as follows:

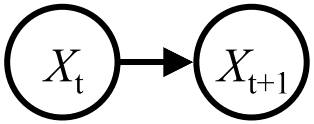
$$\begin{aligned} X_t &= A_{t-1}X_{t-1} + B_{t-1}u_{t-1} + \varepsilon_{t-1} \quad \varepsilon_{t-1} \sim \mathcal{N}(0, Q_{t-1}) \\ Z_t &= C_t X_t + d_t + \delta_t \quad \delta_t \sim \mathcal{N}(0, R_t) \end{aligned}$$

Note: I switched time indexing on u to be in line with typical control community conventions (which is different from the probabilistic robotics book).

Time update

- Assume we have current belief for $X_{t|0:t}$:

$$p(x_t|z_{0:t}, u_{0:t})$$



- Then, after one time step passes:

$$p(x_{t+1}|z_{0:t}, u_{0:t}) = \int_{x_t} p(x_{t+1}, x_t|z_{0:t}, u_{0:t}) dx_t$$

$$\begin{aligned} p(x_{t+1}, x_t|z_{0:t}, u_{0:t}) &= p(x_{t+1}|x_t, z_{0:t}, u_{0:t}) p(x_t|z_{0:t}, u_{0:t}) \\ &= p(x_{t+1}|x_t, u_t) p(x_t|z_{0:t}, u_{0:t}) \end{aligned}$$

Time Update: Finding the joint $p(x_{t+1}, x_t | z_{0:t}, u_{0:t})$

$$
\begin{aligned}
p(x_{t+1}, x_t | z_{0:t}, u_{0:t}) &= p(x_{t+1} | x_t, u_t) p(x_t | z_{0:t}, u_{0:t}) \\
&= \frac{1}{(2\pi)^{n/2} |\Sigma_{t|0:t}|^{1/2}} e^{-\frac{1}{2}(x_t - \mu_{t|0:t})^\top \Sigma_{t|0:t}^{-1} (x_t - \mu_{t|0:t})} \\
&\quad \frac{1}{(2\pi)^{n/2} |Q_t|^{1/2}} e^{-\frac{1}{2}(x_{t+1} - (A_t x_t + B_t u_t))^\top Q_t^{-1} (x_{t+1} - (A_t x_t + B_t u_t))}
\end{aligned}
$$

- Now we can choose to continue by either of
  - (i) mold it into a standard multivariate Gaussian format so we can read of the joint distribution's mean and covariance
  - (ii) observe this is a quadratic form in x_{t} and x_{t+1} in the exponent; the exponent is the only place they appear; hence we know this is a multivariate Gaussian. We directly compute its mean and covariance. [usually simpler!]

Time Update: Finding the joint $p(x_{t+1}, x_t | z_{0:t}, u_{0:t})$

- We follow (ii) and find the means and covariance matrices in

$$(X_{t+1}, X_t)|z_{0:t}, u_{0:t} \sim \mathcal{N}\left(\begin{bmatrix} \mu_{t|0:t} \\ \mu_{t+1|0:t} \end{bmatrix}, \begin{bmatrix} \Sigma_{t|0:t} & \Sigma_{t,t+1|0:t} \\ \Sigma_{t+1,t|0:t} & \Sigma_{t+1|0:t} \end{bmatrix} \right)$$

$\mu_{t|0:t}$ and $\Sigma_{t|0:t}$ are available from previous time step

$$\begin{aligned} \mu_{t+1|0:t} &= \mathrm{E}[X_{t+1}|z_{0:t}, u_{0:t}] \\ &= \mathrm{E}[A_t X_t + B_t u_t + \epsilon_t | z_{0:t}, u_{0:t}] \\ &= A_t \mathrm{E}[X_t|z_{0:t}, u_{0:t}] + B_t u_t + \mathrm{E}[\epsilon_t|z_{0:t}, u_{0:t}] \\ &= A_t \mu_{t|0:t} + B_t u_t \end{aligned}$$

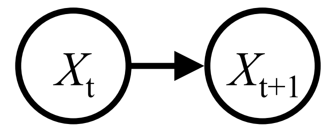
$$\begin{aligned} \mu_{t+1|0:t} &= \mathrm{E}[X_{t+1|0:t}] \\ &= \mathrm{E}[A_t X_{t|0:t} + B_t u_t + \epsilon_{t|0:t}] \\ &= A_t \mathrm{E}[X_{t|0:t}] + B_t u_t + \mathrm{E}[\epsilon_{t|0:t}] \\ &= A_t \mu_{t:0:t} + B_t u_t \end{aligned}$$

$$\begin{aligned} \Sigma_{t+1|0:t} &= \mathrm{E}[(X_{t+1|0:t} - \mu_{t+1|0:t})(X_{t+1|0:t} - \mu_{t+1|0:t})^\top] \\ &= \mathrm{E}[((A_t X_{t|0:t} + B_t u_t + \epsilon_t) - (A_t \mu_{t|0:t} + B_t u_t))((A_t X_{t|0:t} + B_t u_t + \epsilon_t) - (A_t \mu_{t|0:t} + B_t u_t))^\top] \\ &= \mathrm{E}[(A_t(X_{t|0:t} - \mu_{t|0:t}) + \epsilon_t)(A_t(X_{t|0:t} - \mu_{t|0:t}) + \epsilon_t)^\top] \\ &= \mathrm{E}[A_t(X_{t|0:t} - \mu_{t|0:t})(X_{t|0:t} - \mu_{t|0:t})^\top A^\top] + \mathrm{E}[\epsilon_t(A_t(X_{t|0:t} - \mu_{t|0:t}))^\top] + \mathrm{E}[(A_t(X_{t|0:t} - \mu_{t|0:t}))\epsilon_t^\top] + \mathrm{E}[\epsilon_t \epsilon_t^\top] \\ &= A_t \mathrm{E}[(X_{t|0:t} - \mu_{t|0:t})(X_{t|0:t} - \mu_{t|0:t})^\top] A_t^\top + \mathrm{E}[\epsilon_t]\mathrm{E}[(A_t(X_{t|0:t} - \mu_{t|0:t}))^\top] + \mathrm{E}[A_t(X_{t|0:t} - \mu_{t|0:t})]\mathrm{E}[\epsilon_t] + \mathrm{E}[\epsilon_t \epsilon_t^\top] \\ &= A_t \Sigma_{t|0:t} A_t^\top + 0 + 0 + Q_t \end{aligned}$$

$$\begin{aligned} \Sigma_{t,t+1|0:t} &= \mathrm{E}[(X_{t|0:t} - \mu_{t|0:t})(X_{t+1|0:t} - \mu_{t+1|0:t})^\top] \\ &= \Sigma_{t|0:t} A_t^\top \end{aligned}$$

[Exercise: Try to prove each of these without referring to this slide!]

Time Update Recap



- Assume we have

$$
\begin{aligned}
X_{t|0:t} &\sim \mathcal{N}(\mu_{t|0:t}, \Sigma_{t|0:t}) \\
X_{t+1} &= A_t X_t + B_t u_t + \epsilon_t, \\
\epsilon_t &\sim \mathcal{N}(0, Q_t), \ \text{and independent of } x_{0:t}, z_{0:t}, u_{0:t}, \epsilon_{0:t-1}
\end{aligned}
$$

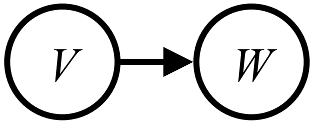
- Then we have

$$
\begin{aligned}
(X_{t|0:t}, X_{t+1|0:t}) &\sim \mathcal{N}\left(\begin{bmatrix} \mu_{t|0:t} \\ \mu_{t+1|0:t} \end{bmatrix}, \begin{bmatrix} \Sigma_{t|0:t} & \Sigma_{t,t+1|0:t} \\ \Sigma_{t+1,t|0:t} & \Sigma_{t+1|0:t} \end{bmatrix} \right) \\
&= \mathcal{N}\left(\begin{bmatrix} \mu_{t|0:t} \\ A_t \mu_{t|0:t} + B_t u_t \end{bmatrix}, \begin{bmatrix} \Sigma_{t|0:t} & \Sigma_{t|0:t} A_t^\top \\ A_t \Sigma_{t|0:t} & A_t \Sigma_{t|0:t} A_t^\top + Q_t \end{bmatrix} \right)
\end{aligned}
$$

- Marginalizing the joint, we immediately get

$$
X_{t+1|0:t} \sim \mathcal{N}\left(A_t \mu_{t|0:t} + B_t u_t, A_t \Sigma_{t|0:t} A_t^\top + Q_t \right)
$$

Generality!



- Assume we have

$$
\begin{aligned}
V &\sim \mathcal{N}(\mu_V, \Sigma_{VV}) \\
W &= AV + b + \epsilon, \\
\epsilon &\sim \mathcal{N}(0, Q), \text{ and independent of } V
\end{aligned}
$$

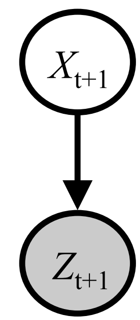
- Then we have

$$
\begin{aligned}
(V, W) &\sim \mathcal{N}\left(\begin{bmatrix} \mu_V \\ \mu_W \end{bmatrix}, \begin{bmatrix} \Sigma_{VV} & \Sigma_{VW} \\ \Sigma_{WV} & \Sigma_{WW} \end{bmatrix} \right) \\
&= \mathcal{N}\left(\begin{bmatrix} \mu_V \\ A_t \mu_V + b \end{bmatrix}, \begin{bmatrix} \Sigma_{VV} & \Sigma_{VV} A^\top \\ A\Sigma_{VV} & A\Sigma_{VV} A^\top + Q \end{bmatrix} \right)
\end{aligned}
$$

- Marginalizing the joint, we immediately get

$$
W \sim \mathcal{N}\left(A\mu_V + v, A\Sigma_{VV} A^\top + Q \right)
$$

Observation update



- Assume we have:

$$
\begin{aligned}
X_{t+1|0:t} &\sim \mathcal{N}\left(\mu_{t+1|0:t}, \Sigma_{t+1|0:t}\right) \\
Z_{t+1} &\sim C_{t+1} X_{t+1} + d_{t+1} + \delta_{t+1} \\
\delta_{t+1} &\sim \mathcal{N}(0, R_t), \text{ and independent of } x_{0:t+1}, z_{0:t}, u_{0:t}, \epsilon_{0:t},
\end{aligned}
$$

- Then:

$$
(X_{t+1|0:t}, Z_{t+1|0:t}) \sim \mathcal{N}\left(\begin{bmatrix} \mu_{t+1|0:t} \\ C_{t+1}\mu_{t+1|0:t} + d \end{bmatrix}, \begin{bmatrix} \Sigma_{t+1|0:t} & \Sigma_{t+1|0:t} C_{t+1}^\top \\ C_{t+1}\Sigma_{t+1|0:t} & C_{t+1}\Sigma_{t+1|0:t} C_{t+1}^\top + R_{t+1} \end{bmatrix}\right)
$$

- And, by conditioning on $Z_{t+1} = z_{t+1}$ (see lecture slides on Gaussians) we readily get:

$$
\begin{aligned}
X_{t+1} | z_{0:t+1}, u_{0:t+1} &= X_{t+1|0:t+1} \\
&\sim \mathcal{N}\left(\mu_{t+1|0:t} + \Sigma_{t+1|0:t} C_{t+1}^\top (C_{t+1}\Sigma_{t+1|0:t} C_{t+1}^\top + R_{t+1})^{-1}(z_{t+1} - (C_{t+1}\mu_{t+1|0:t} + d)),\right. \\
&\qquad \left.\Sigma_{t+1|0:t} - \Sigma_{t+1|0:t} C_{t+1}^\top (C_{t+1}\Sigma_{t+1|0:t} C_{t+1}^\top + R_{t+1})^{-1} C_{t+1}\Sigma_{t+1|0:t}\right)
\end{aligned}
$$

Complete Kalman Filtering Algorithm

- At time 0:

$$X_0 \sim \mathcal{N}(\mu_{0|0}, \Sigma_{0|0})$$

- For t = 1, 2, …
  - Dynamics update:

$$\begin{aligned} \mu_{t+1|0:t} &= A_t \mu_{t|0:t} + B_t u_t \\ \Sigma_{t+1|0:t} &= A_t \Sigma_{t|0:t} A_t^\top + Q_t \end{aligned}$$

  - Measurement update:

$$\begin{aligned} \mu_{t+1|0:t+1} &= \mu_{t+1|0:t} + \Sigma_{t+1|0:t} C_{t+1}^\top (C_{t+1} \Sigma_{t+1|0:t} C_{t+1}^\top + R_{t+1})^{-1} (z_{t+1} - (C_{t+1} \mu_{t+1|0:t} + d)) \\ \Sigma_{t+1|0:t+1} &= \Sigma_{t+1|0:t} - \Sigma_{t+1|0:t} C_{t+1}^\top (C_{t+1} \Sigma_{t+1|0:t} C_{t+1}^\top + R_{t+1})^{-1} C_{t+1} \Sigma_{t+1|0:t} \end{aligned}$$

    - Often written as:

$$\begin{aligned} K_{t+1} &= \Sigma_{t+1|0:t} C_{t+1}^\top (C_{t+1} \Sigma_{t+1|0:t} C_{t+1}^\top + R_{t+1})^{-1} && \text{(Kalman gain)} \\ \mu_{t+1|0:t+1} &= \mu_{t+1|0:t} + K_{t+1}(z_{t+1} - (C_{t+1} \mu_{t+1|0:t} + d)) && \text{“innovation”} \\ \Sigma_{t+1|0:t+1} &= (I - K_{t+1} C_{t+1}) \Sigma_{t+1|0:t} \end{aligned}$$

Kalman Filter Summary

- Highly efficient: Polynomial in measurement dimensionality k and state dimensionality n:

$$O(k^{2.376} + n^2)$$

- Optimal for linear Gaussian systems!

Forthcoming Extensions

- Nonlinear systems
  - Extended Kalman Filter, Unscented Kalman Filter
- Very large systems with sparsity structure
  - Sparse Information Filter
- Very large systems with low-rank structure
  - Ensemble Kalman Filter
- Kalman filtering over SE(3)
- How to estimate A_t, B_t, C_t, Q_t, R_t from data ($z_{0:T}$, $u_{0:T}$)
  - EM algorithm
- How to compute $p(x_t | z_{0:T}, u_{0:T})$ (= smoothing) (note the capital "T")

Things to be aware of that we won't cover

- Square-root Kalman filter --- keeps track of square root of covariance matrices --- equally fast, numerically more stable (bit more complicated conceptually)

- If $A_t = A$, $Q_t = Q$, $C_t = C$, $R_t = R$
  - If system is "observable" then covariances and Kalman gain will converge to steady-state values for t -> 1
    - Can take advantage of this: pre-compute them, only track the mean, which is done by multiplying Kalman gain with "innovation"
  - System is observable if and only if the following holds true: if there were zero noise you could determine the initial state after a finite number of time steps
  - Observable if and only if: rank([C ; CA ; CA^2 ; CA^3 ; … ; CA^{n-1}]) = n
  - Typically if a system is not observable you will want to add a sensor to make it observable

- Kalman filter can also be derived as the (recursively computed) least-squares solutions to a (growing) set of linear equations

Kalman filter property

- If system is observable (=dual of controllable!) then Kalman filter will converge to the true state.
- System is observable if and only if:

  $O = [C ; CA ; CA^2 ; \ldots ; CA^{n-1}]$ is full column rank (1)

Intuition: if no noise, we observe y_0, y_1, … and we have that the unknown initial state x_0 satisfies:

$y_0 = C x_0$

$y_1 = CA x_0$

...

$y_K = CA^K x_0$

This system of equations has a unique solution x_0 iff the matrix $[C; CA; \ldots CA^K]$ has full column rank. B/c any power of a matrix higher than n can be written in terms of lower powers of the same matrix, condition (1) is sufficient to check (i.e., the column rank will not grow anymore after having reached K=n-1).