Markov Decision Processes and Exact Solution Methods:

Value Iteration

Policy Iteration

Linear Programming

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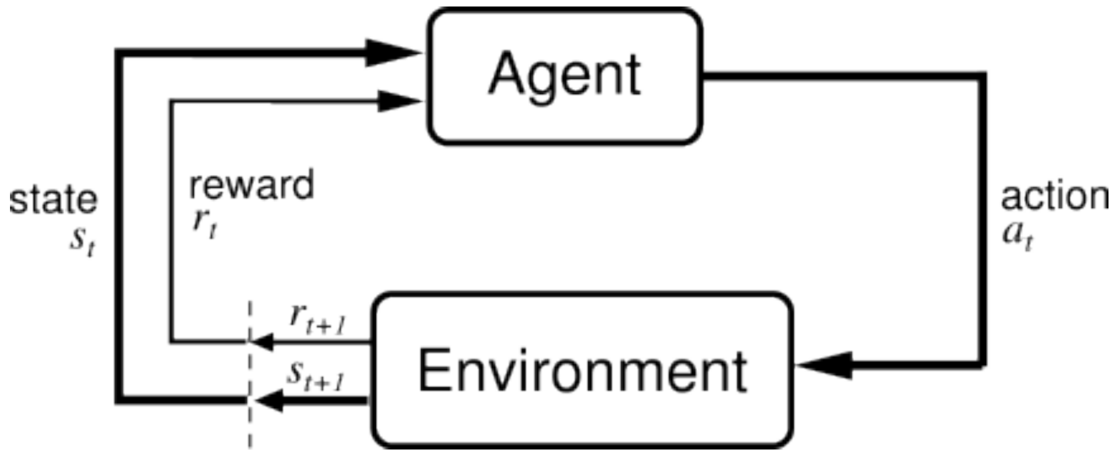
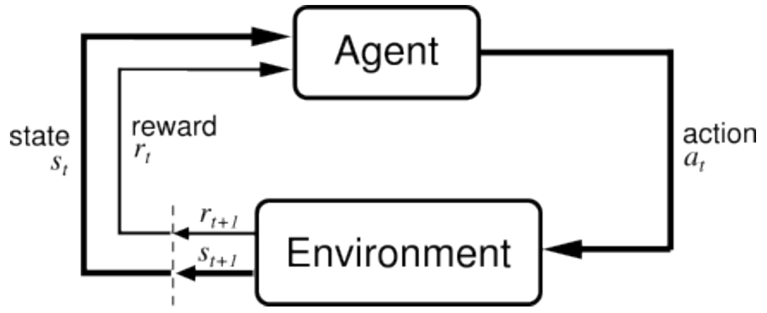
Markov Decision Process



Assumption: agent gets to observe the state

[Drawing from Sutton and Barto, Reinforcement Learning: An Introduction, 1998]

Markov Decision Process (S, A, T, R, γ, H)



Given

- S: set of states
- A: set of actions
- T: S x A x S x {0,1,…,H} → [0,1] $T_t(s,a,s') = P(s_{t+1} = s' \mid s_t = s, a_t = a)$
- R: S x A x S x {0, 1, …, H} → $\mathbb{R}$ $R_t(s,a,s')$ = reward for ($s_{t+1} = s'$, $s_t = s$, $a_t = a$)
- γ in (0,1]: discount factor   H: horizon over which the agent will act

Goal:

- Find π*: S x {0, 1, …, H} → A that maximizes expected sum of rewards, i.e.,

$$\pi^* = \arg\max_{\pi} \mathrm{E}[\sum_{t=0}^{H} \gamma^t R_t(S_t, A_t, S_{t+1}) | \pi]$$

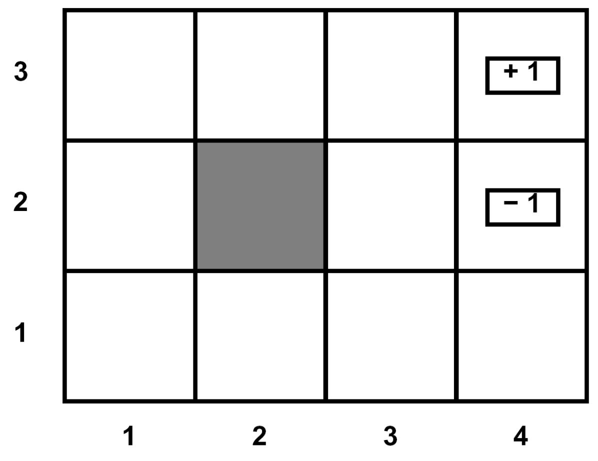
Examples

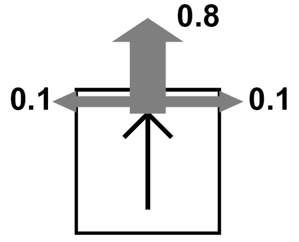
MDP (S, A, T, R, γ, H), goal: $max_{\pi}\mathrm{E}[\sum_{t=0}^{H}\gamma^t R(S_t, A_t, S_{t+1})|\pi]$

- Cleaning robot
- Walking robot
- Pole balancing
- Games: tetris, backgammon
- Server management
- Shortest path problems
- Model for animals, people

Canonical Example: Grid World

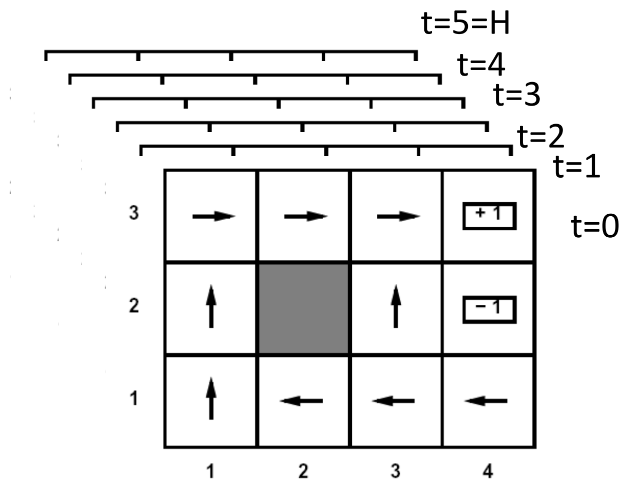
- The agent lives in a grid
- Walls block the agent's path
- The agent's actions do not always go as planned:
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- Big rewards come at the end





Solving MDPs

- In an MDP, we want to find an optimal policy π^*: S x 0:H → A
  - A policy π gives an action for each state for each time



  - An optimal policy maximizes expected sum of rewards
- Contrast: If deterministic, just need an optimal plan, or sequence of actions, from start to a goal

Outline

- Optimal Control

  =

  given an MDP (S, A, T, R, γ, H)

  find the optimal policy π^*

- Exact Methods:
  - ***Value Iteration***
  - Policy Iteration
  - Linear Programming

For now: discrete state-action spaces as they are simpler to get the main concepts across. We will consider continuous spaces later!

[redacted]

Value Iteration

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Algorithm:
        Start with $V_0^*(s) = 0$ for all s.
        For i = 1, … , H
            For all states s in S:
                $V_{i+1}^*(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_i^*(s') \right]$
                $\pi_{i+1}^*(s) \leftarrow \arg\max_{a \in A} \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_i^*(s') \right]$
            This is called a value update or Bellman update/back-up
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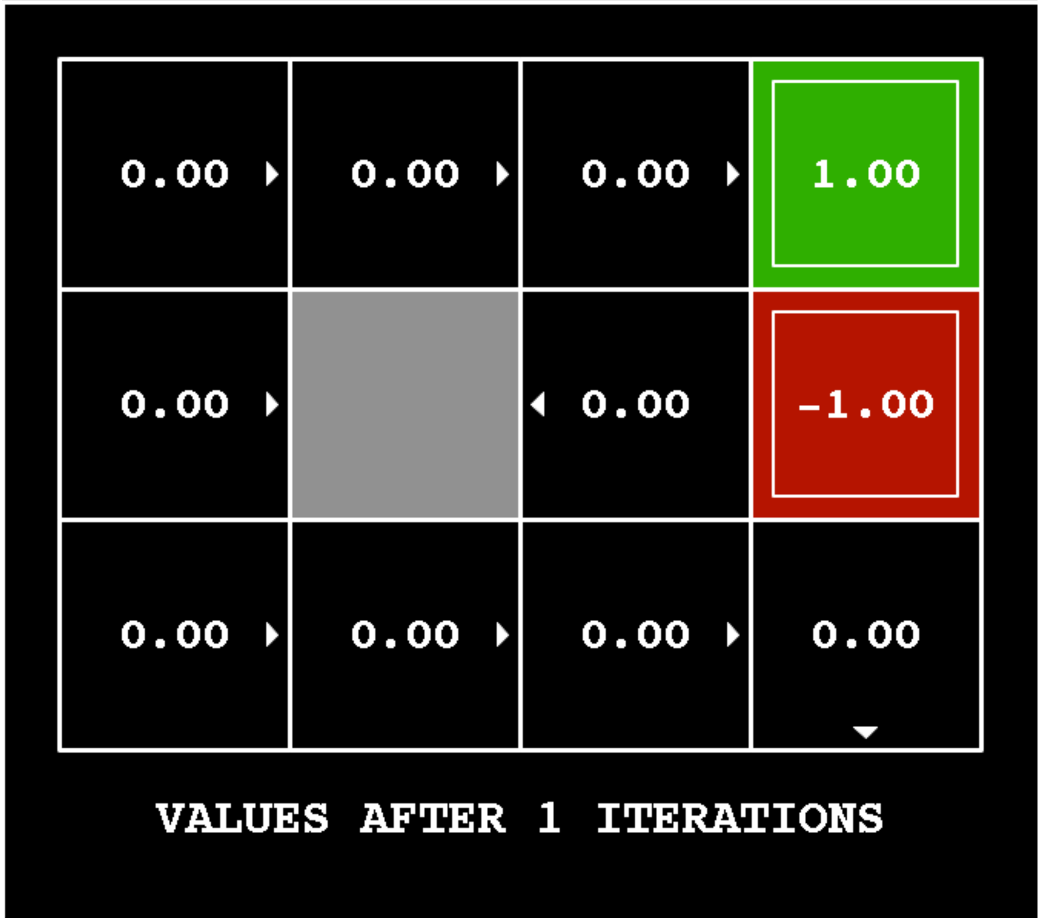
$V_i^*(s)$ = expected sum of rewards accumulated starting from state s, acting optimally for i steps

$\pi_i^*(s)$ = optimal action when in state s and getting to act for i steps

Value Iteration in Gridworld

noise = 0.2, γ =0.9, two terminal states with R = +1 and -1



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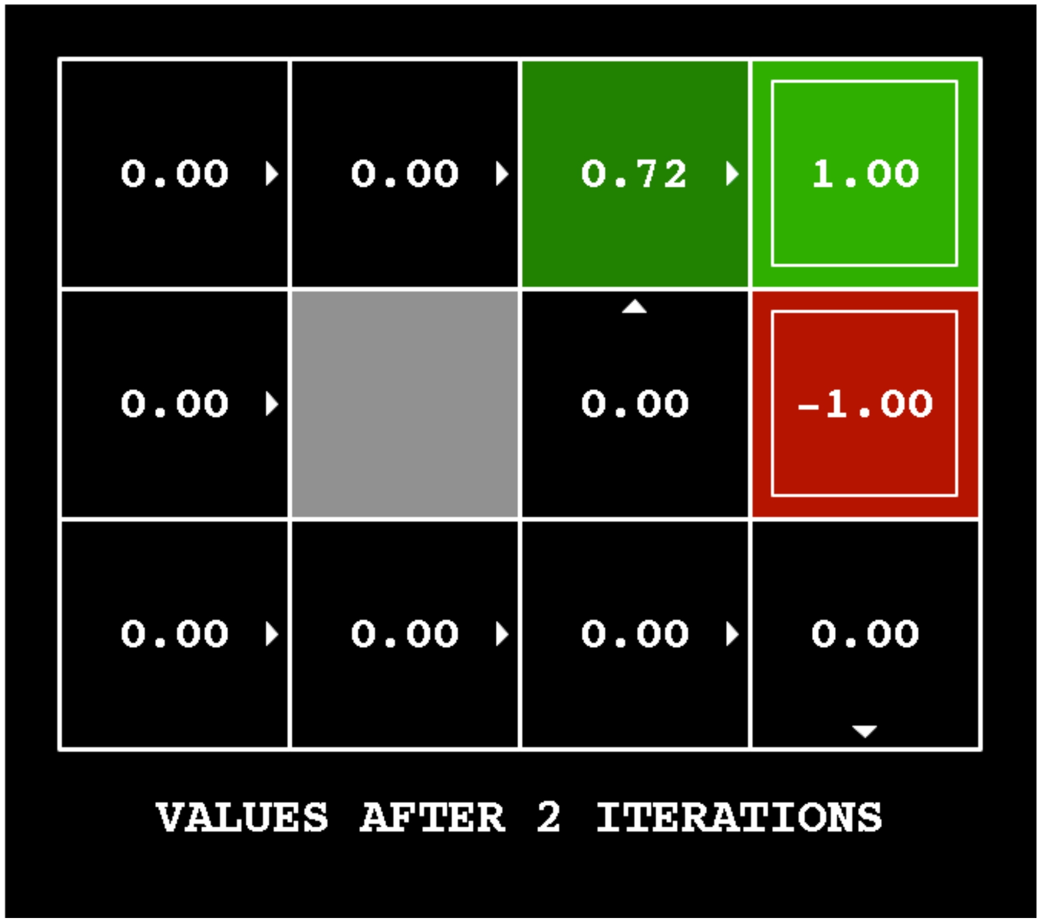

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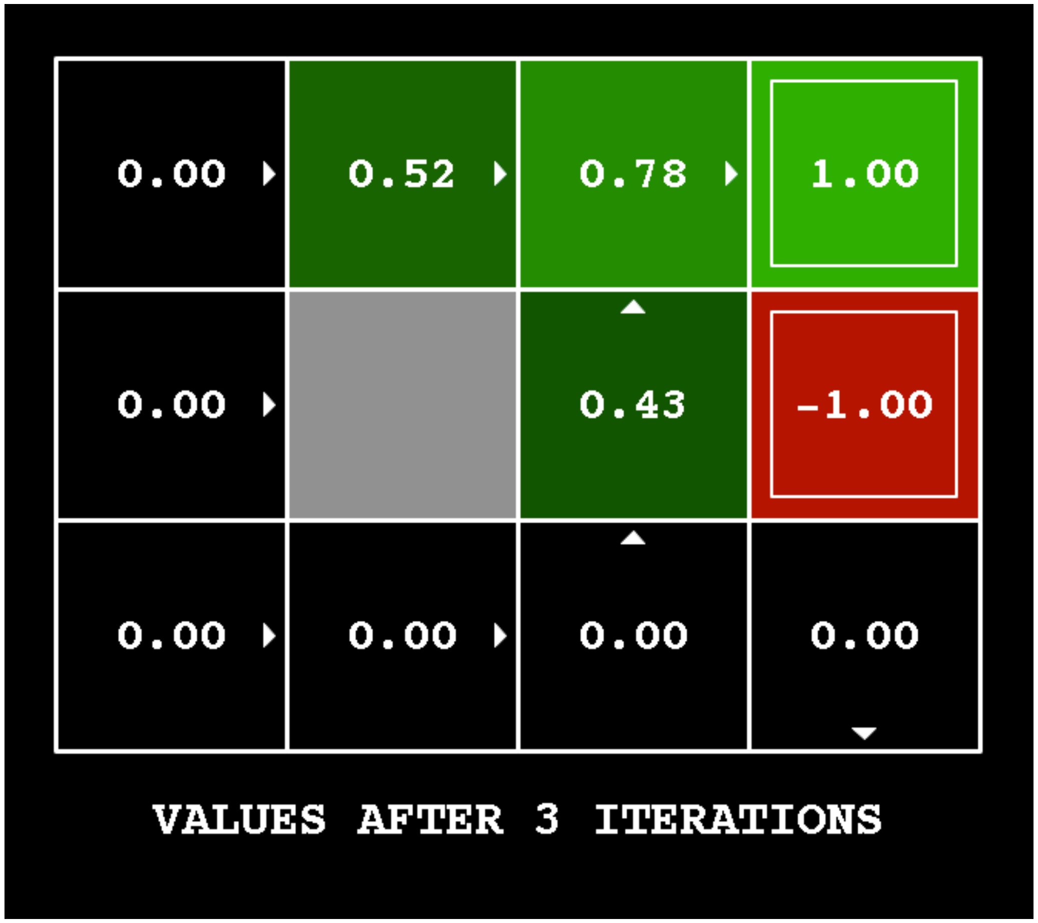

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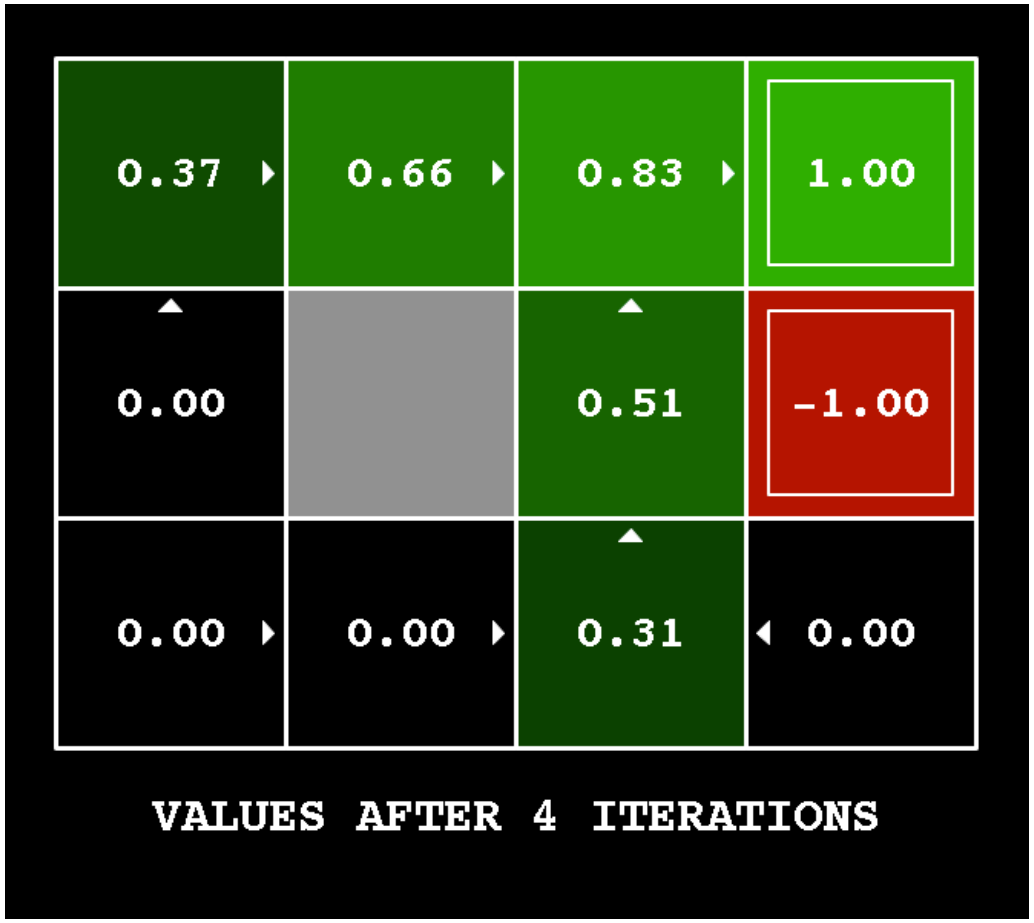

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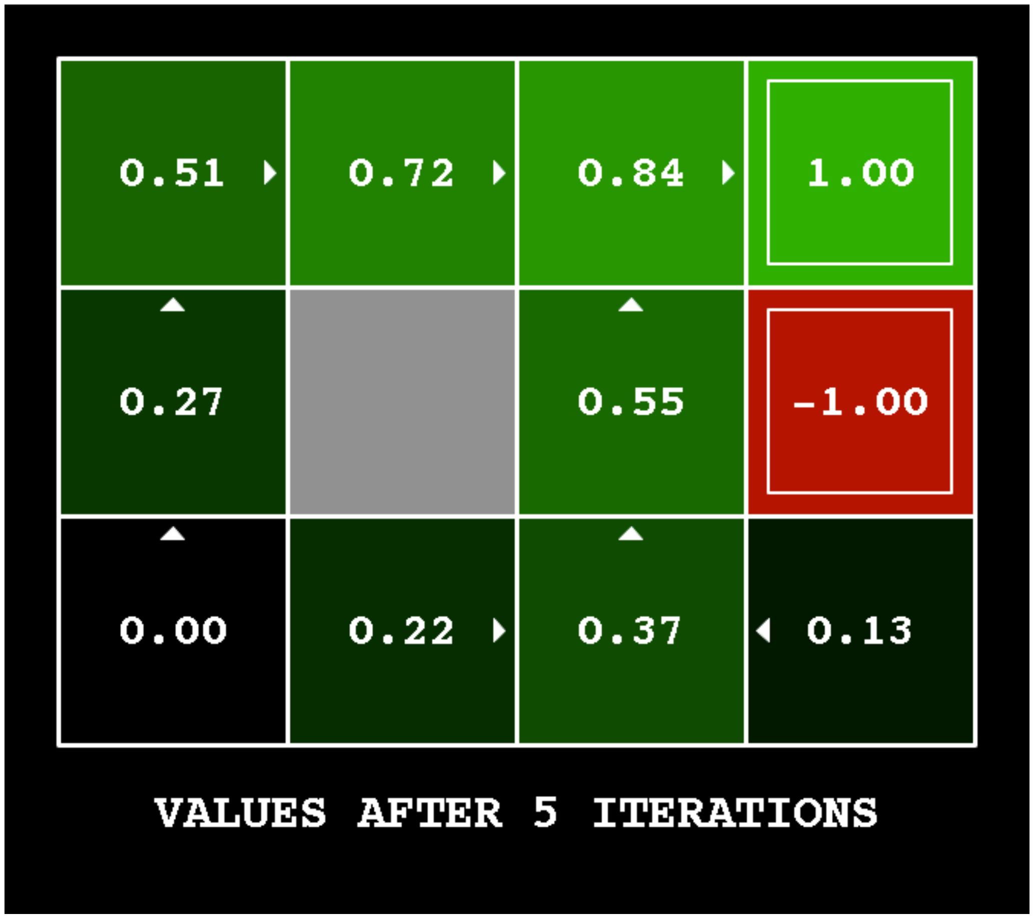

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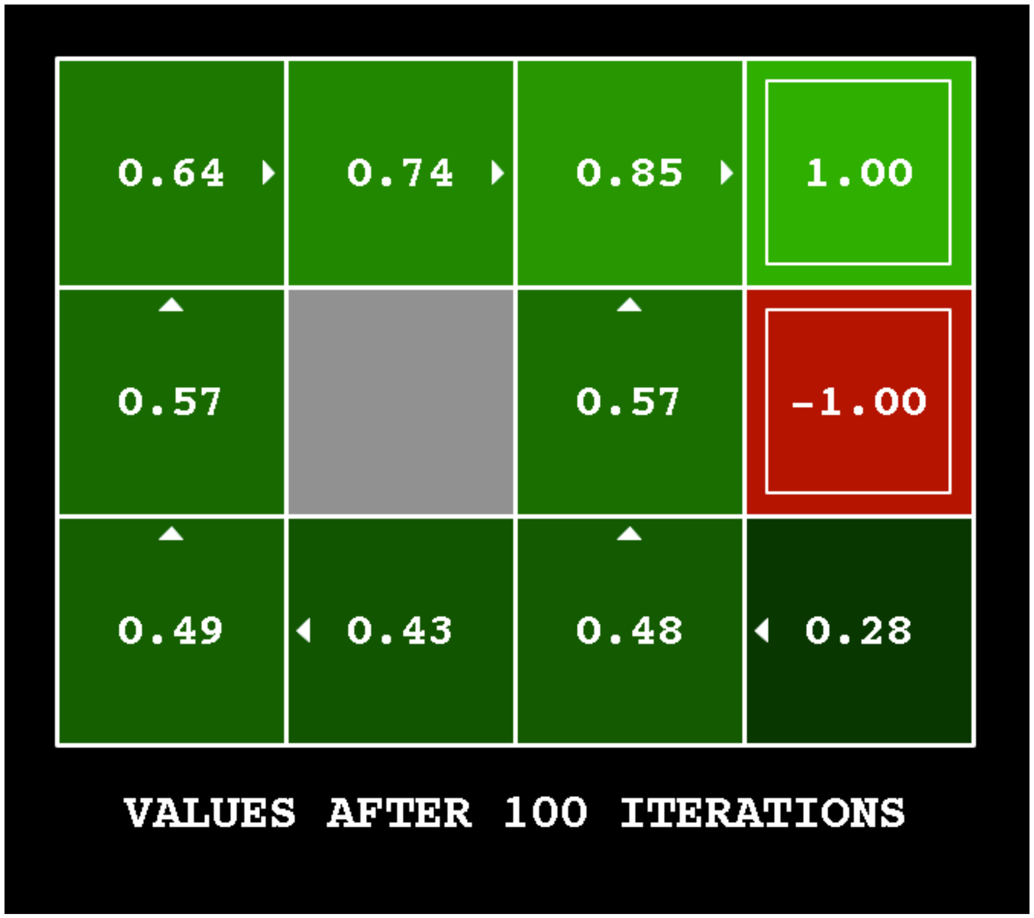

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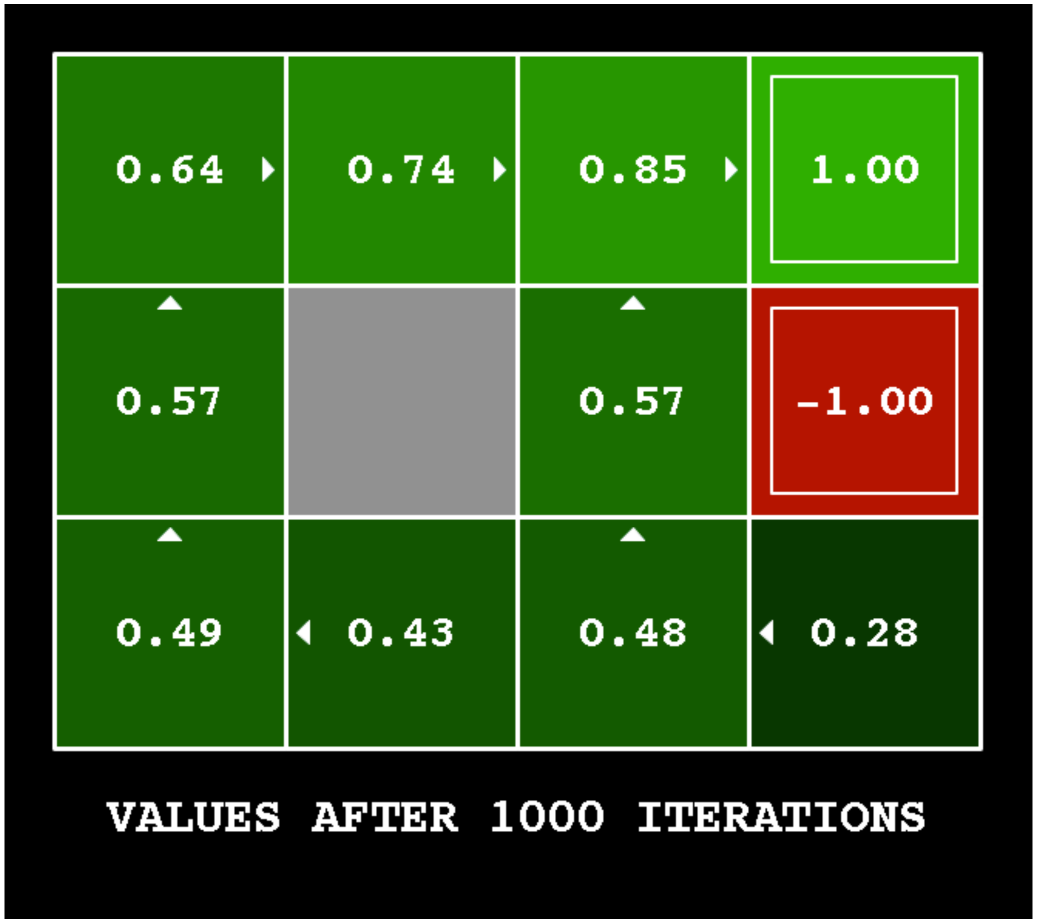


Value Iteration Convergence

Theorem. Value iteration converges. At convergence, we have found the optimal value function V* for the discounted infinite horizon problem, which satisfies the Bellman equations

$$\forall S \in S : \quad V^*(s) = \max_A \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

- Now we know how to act for infinite horizon with discounted rewards!
  - Run value iteration till convergence.
  - This produces V*, which in turn tells us how to act, namely following:

$$\pi^*(s) = \arg\max_{a \in A} \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

- Note: the infinite horizon optimal policy is stationary, i.e., the optimal action at a state s is the same action at all times. (Efficient to store!)

Convergence: Intuition

- $V^*(s)$ = expected sum of rewards accumulated starting from state s, acting optimally for ∞ steps
- $V_H^*(s)$ = expected sum of rewards accumulated starting from state s, acting optimally for H steps

- Additional reward collected over time steps H+1, H+2, …

$$\gamma^{H+1}R(s_{H+1}) + \gamma^{H+2}R(s_{H+2}) + \ldots \leq \gamma^{H+1}R_{max} + \gamma^{H+2}R_{max} + \ldots = \frac{\gamma^{H+1}}{1-\gamma}R_{max}$$

goes to zero as H goes to infinity

Hence $V_H^* \xrightarrow{H\to\infty} V*$

For simplicity of notation in the above it was assumed that rewards are always greater than or equal to zero. If rewards can be negative, a similar argument holds, using max |R| and bounding from both sides.

Convergence and Contractions

- Define the max-norm: $||U|| = \max_s |U(s)|$

- Theorem: For any two approximations U and V

$$||U_{i+1} - V_{i+1}|| \leq \gamma \, ||U_i - V_i||$$

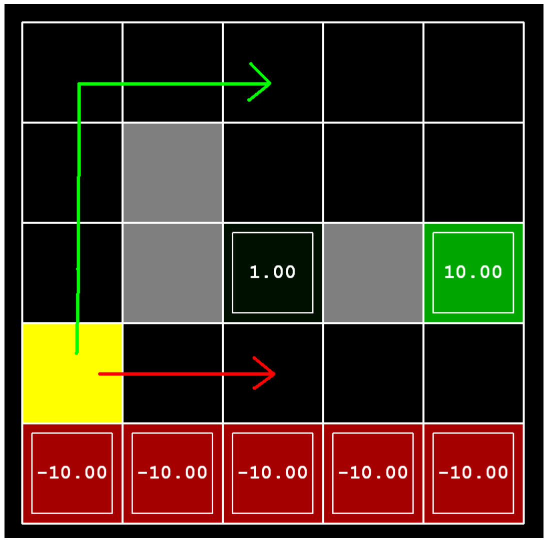
  - I.e., any distinct approximations must get closer to each other, so, in particular, any approximation must get closer to the true U and value iteration converges to a unique, stable, optimal solution

- Theorem:

$$||V_{i+1} - V_i|| < \epsilon, \Rightarrow ||V_{i+1} - V^*|| < 2\epsilon\gamma/(1-\gamma)$$

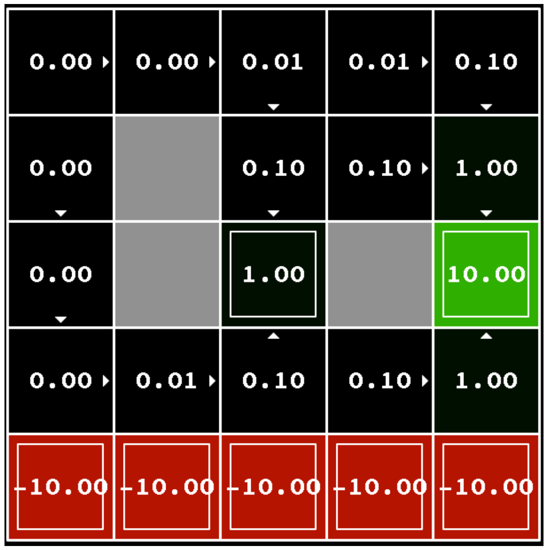
  - I.e. once the change in our approximation is small, it must also be close to correct

Exercise 1: Effect of Discount and Noise



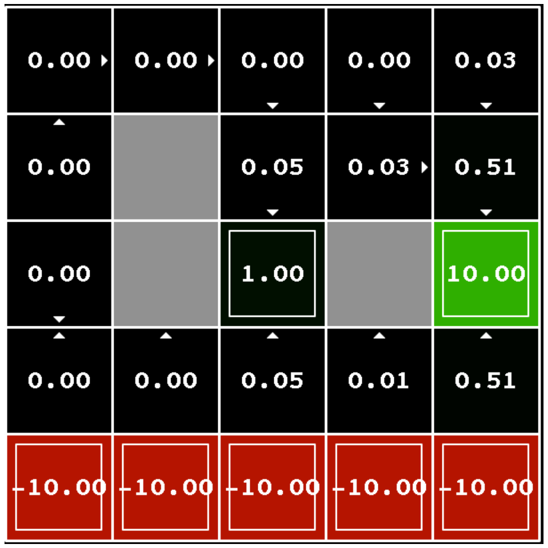
(a) Prefer the close exit (+1), risking the cliff (-10)	(1) γ = 0.1, noise = 0.5
(b) Prefer the close exit (+1), but avoiding the cliff (-10)	(2) γ = 0.99, noise = 0
(c) Prefer the distant exit (+10), risking the cliff (-10)	(3) γ = 0.99, noise = 0.5
(d) Prefer the distant exit (+10), avoiding the cliff (-10)	(4) γ = 0.1, noise = 0

Exercise 1 Solution



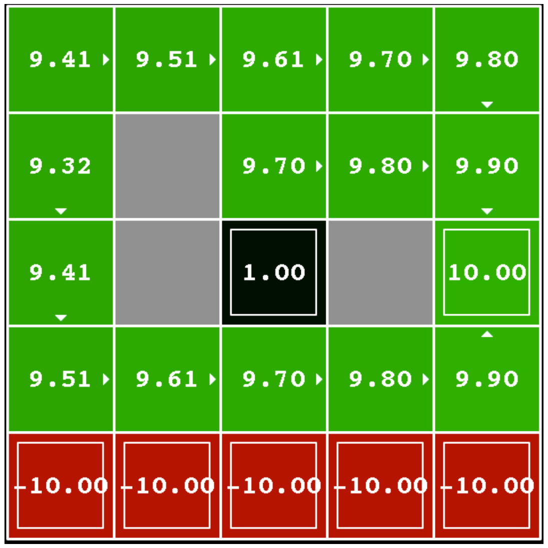
(a) Prefer close exit (+1), risking the cliff (-10) --- (4) $\gamma = 0.1$, noise = 0

Exercise 1 Solution



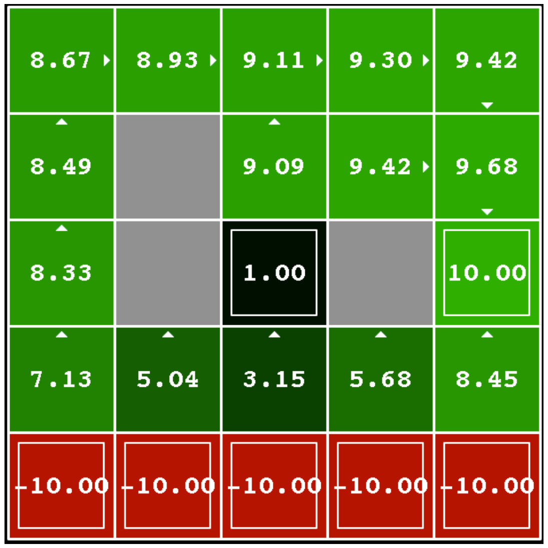
(b) Prefer close exit (+1), avoiding the cliff (-10) --- (1) γ = 0.1, noise = 0.5

Exercise 1 Solution



(c) Prefer distant exit (+1), risking the cliff (-10) --- (2) $\gamma = 0.99$, noise = 0

Exercise 1 Solution



(d) Prefer distant exit (+1), avoid the cliff (-10) --- (3) $\gamma = 0.99$, noise = 0.5

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  find the optimal policy π^*

- Exact Methods:
  - ✓ Value Iteration
  - ***Policy Iteration***
  - Linear Programming

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Policy Evaluation

- Recall value iteration iterates:

$$V_{i+1}^*(s) \leftarrow \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V_i^*(s')]$$

- Policy evaluation:

$$V_{i+1}^{\pi}(s) \leftarrow \sum_{s'} T(s,\pi(s),s')[R(s,\pi(s),s') + \gamma V_i^{\pi}(s')]$$

At convergence:

$$\forall s \quad V^{\pi}(s) = \sum_{s'} T(s,\pi(s),s')[R(s,\pi(s),s') + \gamma V^{\pi}(s')]$$

Exercise 2

Consider a stochastic policy $\mu(a|s)$, where $\mu(a|s)$ is the probability of taking action a when in state s. Which of the following is the correct update to perform policy evaluation for this stochastic policy?

1. $V_{i+1}^{\mu}(s) \leftarrow \max_a \sum_{s'} T(s,a,s')(R(s,a,s') + \gamma V_i^{\mu}(s'))$
2. $V_{i+1}^{\mu}(s) \leftarrow \sum_{s'} \sum_a \mu(a|s) T(s,a,s')(R(s,a,s') + \gamma V_i^{\mu}(s'))$
3. $V_{i+1}^{\mu}(s) \leftarrow \sum_a \mu(a|s) \max_{s'} T(s,a,s')(R(s,a,s') + \gamma V_i^{\mu}(s'))$

[redacted]

Policy Iteration

One iteration of policy iteration:

- Policy evaluation: with fixed current policy π, find values with simplified Bellman updates:
  - Iterate until values converge

$$V_{i+1}^{\pi_k}(s) \leftarrow \sum_{s'} T(s, \pi_k(s), s') \left[R(s, \pi_k(s), s') + \gamma V_i^{\pi_k}(s')\right]$$

- Policy improvement: with fixed utilities, find the best action according to one-step look-ahead

$$\pi_{k+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_k}(s')\right]$$

- Repeat until policy converges
- At convergence: optimal policy; and converges faster under some conditions

Policy Evaluation Revisited

- *Idea 1:* modify Bellman updates

$$V_0^\pi(s) = 0$$

$$V_{i+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_i^\pi(s')]$$

- Idea 2: it is just a linear system, solve with Matlab (or whatever)

  variables: $V^\pi(s)$

  constants: T, R

$$\forall s \quad V^\pi(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^\pi(s')]$$

Policy Iteration Guarantees

Policy Iteration iterates over:

- Policy evaluation: with fixed current policy π, find values with simplified Bellman updates:
  - Iterate until values converge

$$V_{i+1}^{\pi_k}(s) \leftarrow \sum_{s'} T(s,\pi_k(s),s') \left[R(s,\pi_k(s),s') + \gamma V_i^{\pi_k}(s')\right]$$

- Policy improvement: with fixed utilities, find the best action according to one-step look-ahead

$$\pi_{k+1}(s) = \arg\max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^{\pi_k}(s')\right]$$

Theorem. Policy iteration is guaranteed to converge and at convergence, the current policy and its value function are the optimal policy and the optimal value function!

Proof sketch:

(1) *Guarantee to converge*: In every step the policy improves. This means that a given policy can be encountered at most once. This means that after we have iterated as many times as there are different policies, i.e., (number actions)$^{\text{(number states)}}$, we must be done and hence have converged.

(2) *Optimal at convergence*: by definition of convergence, at convergence $\pi_{k+1}(s) = \pi_k(s)$ for all states s. This means $\forall s \;\; V^{\pi_k}(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i^{\pi_k}(s')\right]$
Hence V^{π_k} satisfies the Bellman equation, which means V^{π_k} is equal to the optimal value function V*.

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[redacted]

Infinite Horizon Linear Program

- Recall, at value iteration convergence we have

$$\forall s \in S: \quad V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

- LP formulation to find V^*:

$$\begin{aligned} \min_V \quad & \sum_s \mu_0(s) V(s) \\ \text{s.t.} \quad & \forall s \in S, \forall a \in A: \\ & V(s) \geq \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right] \end{aligned}$$

μ_0 is a probability distribution over S, with $\mu_0(s) > 0$ for all s in S.

Theorem. V^* is the solution to the above LP.

Theorem Proof

Let F be the Bellman operator, i.e., $V^*_{i+1} = F(V_i)$. Then the LP can be written as:

$$\begin{array}{ll} \min_V & \mu_0^\top V \\ \text{s.t.} & V \geq F(V) \end{array}$$

Monotonicity Property: If $U \geq V$ then $F(U) \geq F(V)$.
Hence, if $V \geq F(V)$ then $F(V) \geq F(F(V))$, and by repeated application, $V \geq F(V) \geq F^2V \geq F^3V \geq \ldots \geq F^\infty V = V^*$.
Any feasible solution to the LP must satisfy $V \geq F(V)$, and hence must satisfy $V \geq V^*$. Hence, assuming all entries in μ_0 are positive, V^* is the optimal solution to the LP.

Exercise 3

- How about:

$$\begin{array}{ll} \max_V & \mu_0^\top V \\ \text{s.t.} & V \leq F(V) \end{array}$$

Dual Linear Program

$$\begin{aligned} \max_\lambda \quad & \sum_{s\in S}\sum_{a\in A}\sum_{s'\in S} \lambda(s,a)T(s,a,s')R(s,a,s') \\ \text{s.t.} \quad & \forall s' \in S: \sum_{a'\in A} \lambda(s',a') = \mu_0(s) + \gamma \sum_{s\in S}\sum_{a\in A} \lambda(s,a)T(s,a,s') \end{aligned}$$

- Interpretation:
  - $\lambda(s,a) = \sum_{t=0}^{\infty} \gamma^t P(s_t = s, a_t = a)$
  - Equation 2: ensures that λ has the above meaning
  - Equation 1: maximize expected discounted sum of rewards
- Optimal policy: $\pi^*(s) = \arg\max_a \lambda(s,a)$

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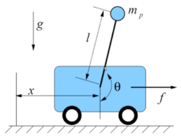
Today and Forthcoming Lectures

- Optimal control: provides general computational approach to tackle control problems.
  - Dynamic programming / Value iteration
    - ✓ Exact methods on discrete state spaces (DONE!)
    - Discretization of continuous state spaces
    - Function approximation
    - Linear systems
    - LQR
    - Extensions to nonlinear settings:
      - Local linearization
      - Differential dynamic programming
  - Optimal Control through Nonlinear Optimization
    - Open-loop
    - Model Predictive Control
  - Examples:



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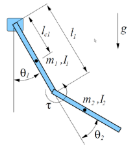

l_{c1}
l_1
g
m_1, I_1
θ_1
τ
m_2, I_2
θ_2