Non-Convex Optimization through Sequential Convex Programming

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Non-Convex Optimization

- Reminder: Convex optimization:

$$\min_x \quad f_0(x)$$
$$\text{s.t.} \quad f_i(x) \leq 0 \quad \forall i$$
$$A(j,:)x - b_j = 0 \quad \forall j$$

with f_i convex

- Non-convex optimization:

$$\min_x \quad g_0(x)$$
$$\text{s.t.} \quad g_i(x) \leq 0 \quad \forall i$$
$$h_j(x) = 0 \quad \forall j$$

with:
g_i non-convex
h_j nonlinear

Sequential Convex Programming

- *To solve:*

$$\begin{aligned} \min_x \quad & g_0(x) \\ \text{s.t.} \quad & g_i(x) \le 0 \quad \forall i \\ & h_j(x) = 0 \quad \forall j \end{aligned} \qquad (1)$$

- *Solve:*

merit function

$$\min_x \quad g_0(x) + \mu \sum_i |g_i(x)|^+ + \mu \sum_j |h_j(x)| \quad = \min_x \quad f_\mu(x) \qquad (2)$$

and increase µ in an outer loop until the two sums equal zero.

- *To solve (2), repeatedly solve the convex program:*

$$\begin{aligned} \min_x \quad & g_0(\bar{x}) + \nabla_x g_0(\bar{x})(x - \bar{x}) + \mu \sum_i |g_i(\bar{x}) + \nabla_x g_i(\bar{x})(x - \bar{x})|^+ + \mu \sum_j |h_j(\bar{x}) + \nabla_x h_j(\bar{x})(x - \bar{x})| \\ \text{s.t.} \quad & \|x - \bar{x}\|_2 \le \varepsilon \end{aligned}$$

(trust region constraint)

$\bar{x}$: current point

Inputs: $\bar{x}, \mu = 1, \varepsilon_0, \alpha \in (0.5, 1), \beta \in (0, 1), t \in (1, \infty)$

While ($\sum_i |g_i(\bar{x})|^+ + \sum_j |h_j(\bar{x})| \geq \delta$ AND $\mu < \mu_{\mathrm{MAX}}$)

$\mu \leftarrow t\mu, \quad \varepsilon \leftarrow \varepsilon_0$ // increase penalty coefficient for constraints; re-init trust region size

While (1) // [2] loop that optimizes

Compute terms of first-order approximations: $g_0(\bar{x}), \nabla_x g_0(\bar{x}), g_i(\bar{x}), \nabla_x g_i(\bar{x}), h_j(\bar{x}), \nabla_x h_j(\bar{x}), \quad \forall i, j$

While (1) // [3] loop that does trust-region size search

Call convex program solver to solve:

$$(\bar{f}_\mu(\bar{x}_{\mathrm{next?}}, \bar{x}_{\mathrm{next?}}) = \min_x \quad g_0(\bar{x}) + \nabla_x g_0(\bar{x})(x - \bar{x}) + \mu \sum_i |g_i(\bar{x}) + \nabla_x g_i(\bar{x})(x - \bar{x})|^+$$

$$+\mu \sum_j |h_j(\bar{x}) + \nabla_x h_j(\bar{x})(x - \bar{x})| \quad \mathrm{s.t.} \quad \|x - \bar{x}\|_2 \leq \varepsilon$$

If $\frac{f_\mu(\bar{x}) - f_\mu(\bar{x}_{\mathrm{next?}})}{\bar{f}_\mu(\bar{x}) - \bar{f}_\mu(\bar{x}_{\mathrm{next?}})} \geq \alpha$

Then shrink trust region:

Else Update $\bar{x} \leftarrow \bar{x}_{\mathrm{next?}}$, Grow trust region: $\varepsilon \leftarrow \varepsilon / \beta$, and Break out of while [3]

If below some threshold, Break out of while [3] and while [2]

Non-Convex Optimization

- Non-convex optimization with convex parts separated:

$$\begin{aligned} \min_x \quad & f_0(x) + g_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0 \quad \forall i \\ & Ax - b = 0 \quad \forall j \\ & g_k(x) \leq 0 \quad \forall k \\ & h_l(x) = 0 \quad \forall l \end{aligned}$$

with:
  f_i convex
  g_k non-convex
  h_l nonlinear

- Retain convex parts and in inner loop solve:

$$\begin{aligned} \min_x \quad & f_0(x) + g_0(x) + \mu \sum_k |g_k(x)|^+ + \mu \sum_l |h_l(x)| \\ \text{s.t.} \quad & f_i(x) \leq 0 \quad \forall i \\ & Ax - b = 0 \quad \forall j \end{aligned}$$