A new similarity measure for covariate shift with applications to nonparametric regression

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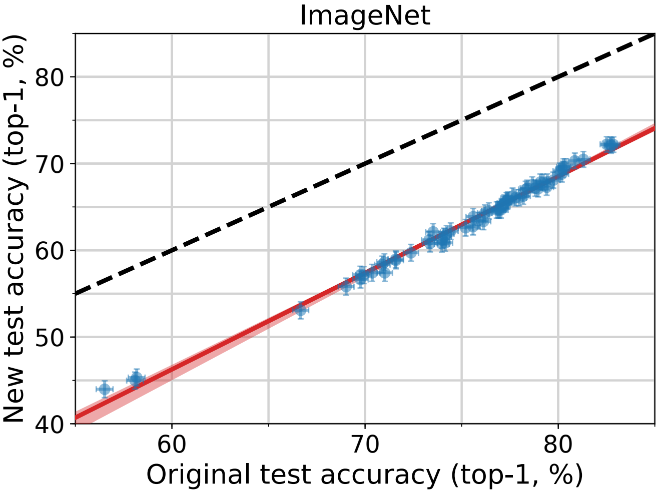
Challenges with distribution shift

Recht, Roelofs, Schmidt, Shankar, 2019



Regression under covariate shift

our work focuses on regression under covariate shift

observational model
we observe a dataset $\{(X_i, Y_i)\}_{i=1}^n$, where

$$Y_i = f^\star(X_i) + \xi_i, \quad i = 1, \dots, n,$$

where $f^\star = \mathbf{E}[Y \mid X = \cdot]$

covariate distribution
covariates are sampled from *source* distribution P and *target* distribution Q:

$$\begin{aligned} &\textit{source} \text{ covariates:} \quad && X_1, \dots, X_{n_P} \overset{\text{i.i.d.}}{\sim} P, \\ &\textit{target} \text{ covariates:} \quad && X_{n_P+1}, \dots, X_{n_P+n_Q} \overset{\text{i.i.d.}}{\sim} Q, \end{aligned} \qquad (n = n_P + n_Q)$$

Similarity measure

we define a measure between two distributions P, Q on metric space $(\mathscr{X}, d)$

similarity measure
for radius $h > 0$, we define

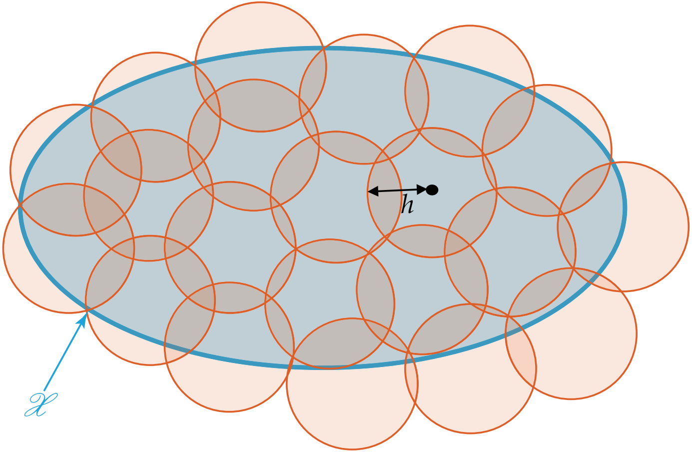
$$\rho_h(P, Q) := \int_{\mathscr{X}} \frac{1}{P(\mathsf{B}(x, h))} \, \mathrm{d}Q(x) = \mathbf{E}_{X \sim Q}\left[\frac{1}{P(\mathsf{B}(X, h))}\right]$$

above, $\mathsf{B}(x, h)$ is closed ball of radius h centered at x

- at fixed $h > 0$, absolute continuity is not required for finite similarity measure
- measure generalizes existing notions of "similarity" for pair (P, Q)
- our results use scaling of mapping $h \mapsto \rho_h(P, Q)$ in limit $h \to 0^+$

Bounds on similarity measure

we bound the similarity measure using covering numbers



covering number $N(h) :=$ minimal number of balls of radius h required to cover $\mathscr{X}$

Bounds on similarity measure

can bound similarity measure by approximating the integral over minimal covers

Proposition

Suppose that for some $h > 0$ there is $\lambda > 0$ such that the mass comparison condition

$$\lambda P(\mathsf{B}(x,h)) \geq Q(\mathsf{B}(x,h))$$

holds for all $x \in \mathscr{X}$. Then, the similarity measure satisfies

$$\rho_h(P,Q) \leq \lambda N(h/2).$$

(note λ can depend on h in claim above)

Consequences of general bound

using previous claim, can bound similarity measure in some situations

examples

- *bounded likelihood ratio:* if $Q \ll P$ and $\frac{\mathrm{d}Q}{\mathrm{d}P}(x) \leq b$ for all x, have $\rho_h(P,Q) \leq b\, N\left(\frac{h}{2}\right)$
- *transfer exponent* (Kpotufe & Martinet, 2018; 2021):
  - pair (P,Q) has (γ, C_γ)-transfer exponent if

$$P(\mathsf{B}(x,h)) \geq C_\gamma h^\gamma Q(\mathsf{B}(x,h)) \quad \text{for all } x \in \mathscr{X}, \text{ all } h > 0. \qquad (\gamma, C_\gamma) \in \mathbf{R}_+ \times (0,1]$$

  - implies similarity measure bound, $\rho_h(P,Q) \leq (h^\gamma C_\gamma)^{-1} N(h/2)$,

(note that $N(h) \lesssim h^{-k}$ as $h \to 0^+$ for compact domains $\mathscr{X} \subset \mathbf{R}^k$)

Assumptions on regression setup

recall our regression setup,

$$Y_i = f^\star(X_i) + \xi_i, \qquad \text{for } i = 1, \dots, n$$

smoothness condition
assume $\mathscr{X} = [0,1]$ and assume that $f^\star$ is L-Lipschitz,

$$f^\star \in \mathscr{F}(L) := \left\{ f : [0,1] \to \mathbf{R} \mid \left| f(x) - f(x') \right| \le L|x - x'| \text{ for any } x, x' \in [0,1] \right\}$$

noise condition
assume the noise variables satisfy (almost surely)

$$\mathbf{E}\left[\xi_i^2 \mid X_i \right] \le \sigma^2, \qquad \text{for } i = 1, \dots, n$$

Classes of covariate shifts

below are families of covariate shift instances based on the map $h \mapsto \rho_h(P, Q)$

families of covariate shifts

- we consider pairs (P, Q) for which (roughly) $\rho_h(P, Q) \lesssim h^{-\alpha}$ as $h \to 0^+$:

$$\mathscr{D}(\alpha, C) := \left\{ (P, Q) \mid \sup_{0<h\leq 1} h^{\alpha} \rho_h(P, Q) \leq C \right\} \qquad (\alpha \geq 1 \text{ and } C \geq 1)$$

- note that $\mathscr{D}(\alpha, C) \subset \mathscr{D}(\alpha', C')$ if $\alpha \leq \alpha'$ and $C \leq C'$

(some additional discussion and extensions in our full paper)

Main result: minimax upper and lower bounds

our minimax results are stated for excess prediction error under Q,

$$\|\hat{f} - f^\star\|_{L^2(Q)}^2 = \mathbf{E}_{X' \sim Q}\left[\left(\hat{f}(X') - f^\star(X')\right)^2\right].$$

Theorem

Suppose $\sigma \geq L$. Let $\alpha \geq 1, C \geq 1$. For a sufficiently large sample size, we have

$$\sup_{(P,Q) \in \mathscr{D}(\alpha,C)} \inf_{\hat{f}} \sup_{f^\star \in \mathscr{F}(L)} \mathbf{E}\|\hat{f} - f^\star\|_{L^2(Q)}^2 \asymp \left\{\left(\frac{n_P}{\sigma^2}\right)^{\frac{3}{2+\alpha}} + \left(\frac{n_Q}{\sigma^2}\right)\right\}^{-\frac{2}{3}}.$$

- when $\alpha > 1$, the worst-case rate (with no access to samples under Q) is $n^{-\frac{2}{2+\alpha}} \gg n^{-\frac{2}{3}}$
- upper bound is achieved by analyzing Nadaraya-Watson estimator under covariate shift
- lower bound is achieved by pair $(P_{\alpha,C}, Q_{\alpha,C}) \in \mathscr{D}(\alpha, C)$ that we construct

Achievable result

achievable result based on classical Nadaraya-Watson estimator

Nadaraya-Watson (NW) estimator
defined pointwise by the local average,

$$\hat{f}(x) := \frac{\sum_{i=1}^{n} Y_i \, \mathsf{1}\{X_i \in \mathsf{B}(x, h_n)\}}{\sum_{i=1}^{n} \mathsf{1}\{X_i \in \mathsf{B}(x, h_n)\}}$$

(above, $h_n > 0$ is a bandwidth parameter)

- the estimator is defined to be zero when denominator is zero
- we establish minimax upper bounds by selecting h_n as a function of $(n_P, n_Q, \sigma^2, L, \alpha, C)$

Lower bound instance

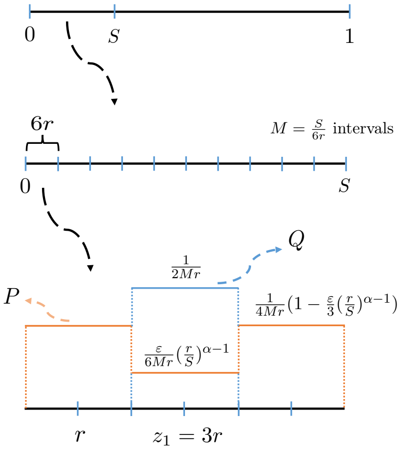


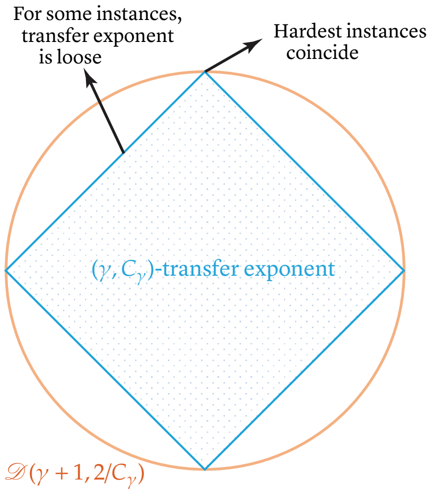
Illustration of lower bound instance

high-level overview

- we construct a hard pair $(P, Q) \in \mathscr{D}(\alpha, C)$
- we construct a hard family of regression functions within $\mathscr{F}(L)$
- we establish our minimax lower bound by combining these two pieces with Fano's inequality and packing-based arguments

Comparison to transfer exponent

introduced by Kpotufe and Martinet, 2018; 2021



our results have consequences for previously proposed notion of transfer exponent

- (P, Q) have (γ, C_γ)-transfer exponent when for all x, h

$$P(\mathsf{B}(x,h)) \geq C_\gamma h^\gamma Q(\mathsf{B}(x,h))$$

- can show if (P, Q) have (γ, C_γ)-transfer exponent, then $(P, Q) \in \mathscr{D}(\gamma+1, 2/C_\gamma)$
- consequently, can obtain upper bounds for instances with known transfer exponent

Conclusions

summary

- we introduce a similarity measure between two probability measures on the same space
- we show that this measure can be bounded easily under natural conditions
- we derive matching minimax upper and lower bounds for nonparametric regression under classes of covariate shifts that are parameterized by the scaling of this measure

additional results (not discussed)

- bounds under more general Hölder-smoothness conditions and additional classes of covariate shifts
- consequences of achievability results for bounded likelihood ratio and transfer exponent