A simplified proof of a Lee-Yang type theorem

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In this short note, we give a simple proof of a Lee-Yang type theorem which appeared in [SS14]. Given an undirected graph $G = (V, E)$, we denote the partition function of the (ferromagnetic) Ising model as

$$Z(G, \beta, z) := \sum_{\sigma : V \rightarrow \{+, -\}} \beta^{d(\sigma)} \prod_{v : \sigma(v) = +} z_v,$$

where $d(\sigma)$ is the number of edges $e = \{i, j\}$ such that $\sigma(i) \neq \sigma(j)$, and $0 < \beta < 1$ is the edge activity. The arguments $z_i$ of the partition function are called vertex activities or fugacities. We then define the operator $D_G$

$$D_G = \sum_{v \in V} z_v \frac{\partial}{\partial z_v},$$

which derives its importance from the fact that the mean magnetization $M(G, \beta, z)$ of the Ising model on $G$ for a given setting of the edge activity and the fugacities can be written as

$$M(G, \beta, z) = \frac{D_G Z(G, \beta, z)}{Z(G, \beta, z)}.$$

The theorem whose proof in this note we simplify is the following:

**Theorem 1** ([SS14]). Let $G = (V, E)$ be a connected undirected graph on $n$ vertices, and assume $0 < \beta < 1$. Then $D_G Z(G, \beta, z) \neq 0$ if for all $v \in V$, $z_v$ is a complex number with absolute value one.

In [SS14], the theorem was proved using a sequence of Asano-type contractions [Asa70], a technique which originated in Asano’s proof of the Lee-Yang theorem [LY52]. The proof we present here completely eschews the Asano contraction in favor of a simpler analytic argument. In our proof we need the following version of the Lee-Yang theorem:

**Theorem 2** ([LY52, Asa70]). Let $G = (V, E)$ be a connected undirected graph on $n$ vertices, and suppose $0 < \beta < 1$. Then $Z(G, \beta, z) \neq 0$ if $|z_v| \geq 1$ for all $v \in V$ and in addition $|z_u| > 1$ for some $u \in V$. By symmetry, the conclusion also holds when $|z_v| \leq 1$ for all $v \in V$ and in addition $|z_u| < 1$ for some $u \in V$.

Observe that given any vertex $u \in V$, we can decompose the partition function as

$$Z(G, \beta, z) = Az_u + B$$

$$A = A(z) = \beta^{\deg(u)} Z(G - \{u\}, \beta, z')$$

$$B = B(z) = Z(G - \{u\}, \beta, z'')$$

where $z' = \left\{ \begin{array}{ll} z_w & \text{when } w \not\sim u \text{ in } G \\ z_w / \beta & \text{when } w \sim u \text{ in } G \end{array} \right.$

$$z'' = \left\{ \begin{array}{ll} z_w & \text{when } w \not\sim u \text{ in } G \\ z_w / \beta & \text{when } w \sim u \text{ in } G \end{array} \right.$$
Neither \(z'\) nor \(z''\) contains \(z_u\) and \(G - \{u\}\) denotes the graph that we obtain from \(G\) by leaving out node \(u\). The Lee-Yang theorem has the following simple consequence, which was also used in [SS14].

**Lemma 3.** If \(G\) is connected, \(0 < \beta < 1\), and all vertex activities have absolute value 1, then \(A\) of eq. (2) is not zero.

**Proof.** Since \(\beta \neq 0\), it is sufficient to prove that \(Z(G - \{u\}, \beta, z') \neq 0\). We observe that the latter is a product of the partition functions of the connected components of \(G - \{u\}\), and furthermore, any neighbor \(w\) of \(u\) in \(G\) in each such component has a vertex activity \(z'_w = z_w/\beta\) with \(|z_w/\beta| = |z_w|/\beta = 1/\beta > 1\). Due to \(G\) being connected, we find such a neighbor \(w\) of \(u\) in all components of \(G - \{u\}\). We apply Theorem 2 to each connected component of \(G - \{u\}\) separately to show that none of the factors is zero.

**Proof of Theorem 1.** Let \(G\) and \(\beta\) be as in the hypotheses of the theorem. Suppose now that there exists a point \(z^0\) such that \(|z^0_v| = 1\) for all \(v\), and \(D_G Z(G, \beta, z^0) = 0\). We will show that this leads to a contradiction. For our subsequent argument it will be helpful to define the univariate polynomial

\[
 f(t) := Z_G(G, \beta, tz^0) \quad \text{where} \quad tz^0 = (t_{z^0_v})_{v \in V}
\]

**Lemma 4.** \(Z(G, \beta, z^0) = 0\)

**Proof.** A comparison of the individual terms gives that \(f'(1) = D_G Z(G, \beta, z^0)\), which is zero by our assumption. From the Lee-Yang theorem we obtain that \(f(t) \neq 0\) when \(|t| \neq 1\), so all zeros of \(f\) must lie on the unit circle. This together with the Gauss-Lucas lemma implies that the derivative of \(f\) cannot disappear on a point of the unit circle unless \(f\) disappears at the same point. Thus, since \(f'(1) = 0\), we get that \(Z(G, \beta, z^0) = f(1) = 0\).

We have that \(f(1 - \epsilon) = f(1) - \epsilon f'(1) \pm O(\epsilon^2) = \pm O(\epsilon^2)\), since the first two terms are zero. Let \(e_u\) be the vertex activity (fugacity) vector with all zero vertex activities except at vertex \(u\) that has activity 1. The key to the proof is to consider the linear perturbation

\[
 Z(G, \beta, (1 - \epsilon)z^0 + \tau e_u)
\]

We show that (3) disappears for some \(\tau \in \mathbb{C}, |\tau| < \epsilon\), in contradiction with the Lee-Yang theorem, since under this assumption all components of \((1 - \epsilon)z^0 + \tau e_u\) have absolute value less than one. By (1):

\[
 Z(G, \beta, (1 - \epsilon)z^0 + \tau e_u) = Z(G, \beta, (1 - \epsilon)z^0) + A((1 - \epsilon)z^0)\tau = \mu_\epsilon + A(z^0)\tau + \nu_\epsilon \tau
\]

Here \(\mu_\epsilon = f(1 - \epsilon) = \pm O(\epsilon^2)\), and \(\nu_\epsilon = A((1 - \epsilon)z^0) - A(z^0) = \pm O(\epsilon)\) by the analyticity of the function \(A\). Recall that \(A(z^0) \neq 0\) by Lemma 3. Then expression (3) disappears at \(\tau = -\frac{\mu_\epsilon}{A(z^0) + \nu_\epsilon}\), and also \(|\tau| < \epsilon\) if \(\epsilon\) is sufficiently small.

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