**CS270 Algorithms & Data Structures** **Spring 2008**

# Lecture 3-4: Eigenvalues and cuts:supplementary. : 1.30.08

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**Disclaimer**: *These are rough notes, with some exercises.*

A student requested a summary of the definitions and references where they were stated more precisely.

One book is "Spectral Graph Theory (CBMS Regional Conference Series in Mathematics, No. 92) (Cbms Regional Conference Series in Mathematics) (Paperback) by Fan R. K. Chung (Author)."

So, some of the definitions that I used most were as follows.

## 3.1 Definitions of vector optimization problems.

The Raleigh quotient of a graph is defined as

$$\rho(G) = \min_{x_1,\ldots,x_n} \frac{\sum_{e=(i,j)} (x_i - x_j)^2}{\sum_{(i,j)} (x_i - x_j)^2}$$

After this, I made an assumption that I am working with graphs where all nodes have the same degree (to avoid the notation and subtleties introduced by variation in degree which I feel is a whole 'nother subject.)

So, the following definitions, I use by may not be fully standard.

$$\lambda(G) = \min_{x_1,\ldots,x_n : \sum_i x_i = 0} \frac{\sum_{e=(i,j)} (x_i - x_j)^2}{\sum_{(i,j)} (x_i - x_j)^2}.$$

We noted that

$$\lambda(G) = n\rho(G).$$

Finally, we defined $\lambda_2$ as the second largest eigenvalue of the transition matrix for a random walk on the graph with stationary probability 1/2. For different developements, use a matrix called the Laplacian. See, for example, Arora's notes (http://www.cs.princeton.edu/courses/archive/fall02/cs597D/lec6.ps).

We noted that, for uniform degree $d$ graphs that

$$1 - \lambda_2^2 = \lambda(G)/2d.$$

In terms of due diligence; we recall that random walks is really a behavior on edges. The true behavior is that it is uniform over edges and thus, for example, high degree nodes are hit more often than low degree nodes. This complicates things and changes the definitions of, in particular, the graph that one analyzes to figure out mixing rates. It is an aspect that I believe is less interesting than the aspects that we cover. Moreover, it is messier and beyond the scope of this course.

## 3.2 Cut quality definitions.

We define the sparsity, $\mathcal{S}$, of a cut as

$$\mathcal{S}(S, \overline{S}) = \frac{c(S, \overline{S})}{|S||\overline{S}|}.$$

The sparsity of a graph is the smallest sparsity of any cut.

We define the edge expansion, $\alpha$, of a cut as

$$\alpha(S, \overline{S}) = \frac{c(S, \overline{S})}{\min(|S|, |\overline{S}|)}.$$

We have the relationship between the two

$$n\mathcal{S}(S, \overline{S})/2 \leq \alpha(S, \overline{S}) \leq n\mathcal{S}(S, \overline{S})$$

Before leaving this, we note the definition of conductance $\Phi(G)$ to be as follows.

$$\Phi(G) = min_S \frac{c(S, S)}{vol(S)vol(\overline{S})},$$

where $vol(S)$ is the number of edges induced on the subset $S$.

These definitions can be extended to weighted graphs in the natural manner.