Fun with planar graphs.

CS270 Combinatorial Algorithms & Data Structures **Spring 2003**

Lecture 1 Fun with planar graphs. : 2.12.03

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The notes have been remembered or inspired by or taken from various sources. Apologies to those who were plagiarized and/or not cited. Some of the works include [2, 3, 1].

1.1 Warm Up.

Question: Is $\sqrt{2}$ irrational?

Yes.

Question: Can you prove it?

Well. I know how?

Question: What's a rational number?

Can be expressed a/b.

Question: How do you start a proof by contradiction?

$\sqrt{2} = a/b$

Question: Can you apply the "well-ordering principle?"

a/b is reduced.

Sub Question: Is well ordering principle related to induction?

Question: How to proceed?

$2 = a^2/b^2$. Then $a^2 = 2b^2$. a is even, since any square with a factor of two must have another.

Question: How to proceed?

$a = 2c$. $2b^2 = 4c^2$. b is even.

Question: Contradicts what?

a/b is reduced.

Question: Does this work for any non-square integer?

Question: What step doesn't follow for a perfect square?

1.2 Planar maps.

Question: Draw a map!

Question: Consider coloring it so that no two countries that share a non-trivial border are different colors. (Sharing a corner is OK)? Use the fewest?

Question: Can you make a map that requires 3 colors? Why does it?

Question: Can you make a map that requires 3 colors, where no three countries all touch each other?

Question: Can you make a map that requires 4 colors?

Question: Can you make a map that requires 5 colors?

No.

Question: Can you prove it?

No.

Question: Can you make a map that requires 6 colors?

No.

Question: Can you prove it?

We will do that now.

Question: Have you heard of graph coloring?

I guess so.

Question: What is a graph?

Set of nodes, set of node pairs, "edges".

Sub Question: Connected? Node a and b are connected, if there is a sequence of edges that.

Sub Question: Connected Components?

Every node in set is connected to every other. Paritions the set of nodes, since connected is transitive.

Question: Graph Coloring?

Color nodes, so no edge is singly colored. Use fewest.

Question: Graphs and maps, how do they correspond?

Intersection points of borders can be nodes. Edges can be borders between nodes. Planar since borders don't cross. We call the countries "faces." We also call the "outside face" or infinite region of the place a face. This is a planar graph; a graph that can be (is) drawn in the plane with no edges crossing.

Question: Graph coloring and map coloring? We are coloring countries not intersection points!

Make node for country. Make edges between countries. with common borders. Also planar. The planar dual! Color this.

Question: The dual of the dual?

The original, if the original is connected.

Question: Is the dual connected?

The dual is always connected since there is always a way to cross edges to get to the exterior face.

Question: Can you draw a planar graph with five countries that all touch each other?

No. Why?

1.3 Euler's formula?

Question: Graphs? If connected and no cycles, how many edges?

If connected, there is a path between every pair. Need $n - 1$, since every edge can reduce number of components by 1, need to reduce from n to 1.

If more than $n - 1$, an edge must be added at some point between two nodes that are already connected. We can then reconstruct a cycle?

Question: What does a simple closed curve do to the plane?

Divides it into two "connected" regions; Any path between points in different regions intersects the curve, for any pair of points in the same region, there exists a path between them that does not intersect the curve. This is called the jordan curve theorem.

Question: What does a cycle in a planar dual correspond to?

A jordan curve surrounding one or more nodes in the primal.

Question: Take a spanning tree of a planar graph? How many edges?

$n - 1$.

Question: Take the planar dual? Take the set of dual edges that correspond to nontree edges? What is it?

A tree. It is connected since there is no cycle in the primal. The ensures that there is a path from any face to the exterior face by crossing edges, since there is no Jordan curve.

Then, assume it is not a tree, then there would be a cycle in the dual graph edges. This corresponds to a simple cycle in the plane. Thus, by the Jordan curve theorem there are two regions. Since, this cycle corresponds to non-tree edges of the primal, no edge in the primal tree crosses this Jordan curve. This contradicts the idea that the primal tree was connected. That is the nodes in the cycle are separated.

Question: How many edges in the dual formula?

$|F| - 1$.

Question: So what?

$|E| = |V| + |F| - 2$. Euler's Formula.

(Another proofs. Start with a tree. One face, add an edge, add a face. Etc. Nineteen proofs. Search for "Eppstein, Nineteen proofs of Euler's Formula" or http://www.ics.uci.edu/ eppstein/junkyard/euler/.)

Question: Can five countries touch each other?

Hmmm.. well there are lots of edges.

Question: Can it have lots of edges?

Hmmm..

Question: How many faces is each edge next to?

Exactly 2 (if we allow the infinite face to be adjacent to the same edge twice.)

Question: Thus, how many (edge,face) adjacencies are there?

$2|E|$.

Question: How many edges is each face incident to? (Simple connected graph. More than three nodes.)

At least 3, if simple connected graph.

Question: Thus, how many (edge,face) adjacencies are there.

At least $3|F|$.

Question: How do the number of (edge,face) adjacencies compare to the number of (face,edge) adjacencies.

They are the same.

Question: OK, now what?

$$3|F| \leq \text{ number of (face,edge) adjacencies} = \text{number of (edge,face) adjacencies } \leq 2|E|. \tag{1.1}$$

Question: Can we solve for $|F|$ in Euler's formula?

Well, I think so. $|F| = |E| - |V| + 2$

Question: And, plug in to 1.1?

Hmmm. $3|E| - 3|V| + 6 \leq 2|E|$.

Question: Isolate $|E|$ and get an upper bound on $|E|$.

$|E| \leq 3|V| - 6$.

Question: K_5 is planar?

No. $10 > 9$. Can't be planar.

Question: Does deletion preserve planarity?

Yes.

Question: Does contraction of degree two vertices preserve planarity?

Yes.

Question: K_5 does not appear even after deleting and contracting degree two vertices. Right?

Right.

Exercise 1: Prove that $K_{3,3}$ in not planar. $K_{3,3}$ consists of six nodes

divided into two groups of 3. Each node has an edge to every node in the other group.

Question: Hmmm. Average degree is low.

Question: Color degree Δ graph, with $\Delta + 1$ colors?

Sort by degree, low to high. Start coloring right to left. Always one color available.

Question: Inductively?

Inductive algorithm. Remove degree delta node. Color remaining graph with less than $\Delta + 1$ colors. Add degree delta node and color it. One of the $\Delta + 1$ colors will be available by the pigeonhole principle.

Question: Average degree of planar graph?

Less than 6. The sum of the degrees is equal to $2|E|$, since each edge appears twice. This is less than $6n - 6$. Thus, average degree is less than 6.

Question: Six color a planar graph?

Find a degree 5 node. Remove it. Color remaining graph inductively with 6 colors or less. Then we can extend the color to this graph upon adding it back.

Question: Five color a planar graph? Where to start?

Find a degree 5 node. All five colors appear around it. If not, done. Think of the planar embedding. Label 1,2,3,4,5. Nodes, v_1, v_2, v_3, v_4, v_5.

Question: Can we change colors a bit?

Switch 1 and 3 everywhere in the previous coloring.

Question: Does this help?

No.

Question: Connected components of nodes colored 1 and 3? Can we switch the colors in just one connected component?

Yes. Edges inside are ok since you are just swapping the (different) colors of the endpoints. Edges out are to 2,4, and 5 colored nodes.

Question: If v_1 and v_3 are in different connected components, what to do?

Switch colors for v_1's component. v_1 is now colored 3, and we can use color 3.

Question: If not? What must happen?

There is a path of consisting of nodes colored either 1 or 3 between v_1 and v_3.

Question: Let's try for 2 and 4. Either works or what?

There is a path of 2 or 4 colored nodes between v_2 and v_4.

Edges must cross or a node has two colors?

Question: Planar drawings are tricky. If both paths above exist, can you see K_5?

Kind of. Assume graph is triangulated. Then, nodes 1 through 4 and the center node form K_5 in the following sense; that is there are disjoint paths in the graph that connect all of these nodes in the graph.

Thus, both paths cannot exists, and one can fixup the previous coloring to enable one to color this vertex.

Question: Four colors?

Well. No. But, vaguely similiar. We said a degree five vertex had to exist, and then showed that any coloring of the rest can be extended to this. They show that there is one of some set of subgraphs(our case, degree five vertex) that exist (manipulating Euler's formula) and then show how to extend any coloring from rest of graph to subgraph. The "computer part" of the proof is in manipulating Kempe chains to prove that a coloring can be extended. Also, the other part, i.e., proving that one of the subgraphs appears is quite long and has been translated to a language that can be verified by computer. See http://www.math.gatech.edu/ thomas/FC/fourcolor.html for further discussion.

Exercise 2: An exercise. Discharging.

Prove that every triangulated planar graph (every face is a triangle), either contains a vertex of degree 1, 2,3, 4, or two degree five nodes that are connected or a degree five and a degree six node which are connected.

Assume there are no degree 1,2,3, or 4 nodes.

Exercise 2.1: Place a charge on each node v of value $(6 - d(v))$. What is the sum of the charges on all the nodes? (Hint: Euler's formula and $|F| = 2|E|/3$ when all faces are triangles.)

Exercise 2.2: What is the charge of a degree 5 node? A degree 6 node?

Exercise 2.3: Move $1/5$ of 1 charge from each degree five node to each of its negatively charged neighbors. If there is a degree five vertex with positive remaining charge, are we done? Why or why not?

Exercise 2.4: If no degree five vertex has positive charge, is there a node with positive charge? What must its degree be?

Exercise 2.5: For a degree seven vertex with positive charge, how many neighbors have degree five?

Exercise 2.6: If the graph is triangulated, must two degree five neighbors be adjacent? Are we done yet?

Exercise 3: Show that for any planar graph with a node of degree 4, that it is not a smallest counterexample to the four color theorem.

References

[1] D. Eppstein. www.ics.uci.edu/ eppstein/junkyard/euler/.

[2] Various. http://inst.eecs.berkeley.edu/ cs70/archives.html.

[3] R. Wilson. *Four Colors Suffice*. Princeton University Press, 2002.