CS270 Algorithms & Data Structures **Spring 2008**

Lecture 11 linear programming (continued). : 2.19.08

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Disclaimer: *These are rough notes, with some exercises.*

11.1 Duality: continued.

Question: Is the value of the dual a lower bound on the value of the primal, in general?

Let's think of a feasible solution x. And consider $b - Ax$, which is positive. Let's multiply this vector by a feasible y of value yb.

We get $yb - yAx$ which must now be a positive value. But $yA \geq c$, and $yAx \geq cx$. That is, $yb - cx$ must be positive. That is the value of any feasible x, cx is at most as large as the value of any feasible y, yb.

This is called weak duality. (Working things out, this is a similar argument to showing that the minimum cut is a lower bound on the maximum flow.)

Question: Huh?

Ok, so a feasible dual solution is simply adding the equations in the primal so that the resulting form has larger coefficients than the optimization function. The resulting right hand side must then be an upper bound on the value of the optimization function for x since any positive linear combination of the equations must be satisfied.

That is, the same as the proof above.

Question: Adding equations? Something to do with Gaussian elimination?

First, note by Gaussian elimination that one can find some way to express the optimization function that is a linear combination of the equations. In fact, there are many such ways since the set of inequalities is larger than the set of variables.

The challenge is to find a positive linear combination. That "dominates" it as little as possible.

Question: How do we show that it is equal?

Let's look back at the simplex method. At the end, we have

$$\begin{array}{rcl} A_B^{-1}A| & A_B^{-1} & |A_B^{-1}b \\ -c + c_B A_B^{-1}A| & 0 + c_B A_B^{-1} & |z \end{array}$$

Note, this notation varies a bit from before and needs some explanation. A is the matrix in the original max problem. A_B is the subset of columns (variables in the original program) from A and the added columns (slack variables) that form a basis. c_B is c on the values associated with original variables and 0 for values associated with the slack variables.

At optimality the bottom row is all positive. Notice that the number of entries after the bar is equal to the number of equations in the original maximization primal.

Let's take the bottom row after this point to be a possible dual solution to the dual program of A_0. Note that this is $y = c_B A_B^{-1} A$.

Question: Is y it positive?

Sure, otherwise simplex would have continued.

Question: What is the dual value, yb?

At this point, $x_B = A_B^{-1} b$ and all other x are zero.

So, $yb = -c_B A_B^{-1} I b = -c_B x_B$. Since the only nonzero x's in this solution are the x_B's. This says $yb = cx$. That is, yb is equal to the value of the primal, and y is positive.

Question: Is $yA \geq c$, though?

Again, $yA - c = c_B A_B^{-1} A - c$. This is the bottom row of the simplex algorithm tableau. Which at termination is all positive. Thus,

$$yA \geq c.$$

That is, the program is feasible.

This is the duality theorem.

Exercise 1: Show that $z = cx = by$, where x is defined by the last column (i.e., the last column corresponds to the values of the basic variables, the non-basic variables taking value 0) in the tableau, and y is defined as above (the portion of the last row under the slack variables.

Question: As the simplex algorithm goes, how is the dual solution evolving?

Well, it is not feasible. But has a better value than than it should.

Question: And the primal?

Well, it is always feasible, but is not optimal.

Question: Complementary slackness?

The only non-zero dual values correspond to non-basic slack variables (or zero valued slack variables.) Which means they correspond to tight equations.

Question: And vice versa?

Sure, the basic variables correspond to columns that have zero dual feasibity gap. That is, where the dual inequalities are tight.

Question: Complementary slackness and optimality?

When one has a feasible primal dual pair, that obey complementary slackness, both linear programs are optimal. This follows from examining the expression $y(b - Ax)$ is 0 since y is nonzero only on rows where $b_i - a_i x = 0$ and $yb - yAx = yb - cx$ since x is nonzero only on rows where $ya_j^T = c_j$. That is the duality gap is 0.

Question: Maximum flow and complementary slackness?

The dual (cut variables) are positive only on saturated edges. The flow is positive only on shortest paths.

Question: Diet problem?

$$Ax \geq b$$

$$\min cx$$

Each constraint corresponds to a vitamin requirement. Each column corresponds to a type of food. The c vector corresponds to its cost.

Question: Dual?

$$y^T A \leq c$$

$$\max by$$

In the dual, vitamin seller, the constraint on food cost of vitamin combination. Vitamins should be cheaper. Maximize the cost so that this is true.

Question: Complementary slackness.

In a solution, only buy foods that are cost effective; provide cost effective vitamins. Only charge for vitamins that consumer is tight on.

Question: Games?

Given a game as a matrix, A. Two players; a row player and a column player. The row player plays i, and the column player plays j. The row player pays a_{ij} to the column player.

Question: E.g?

The matrix for Roshambo, "Rock, paper, scisssors."

$$0 - 1\ 1$$

$$-1\ 0 - 1$$

$$1 - 1\ 0$$

The strategies 1,2,3; Rock, paper,scissors.

Optimal strategies; $1/3, 1/3, 1/3$ for each player.

Question: As linear program?

The column player wishes to make sure row player gets no good response. That is,

$Ax \geq z$ $\max z$

The row player wishes to make sure the column player gets no good response.

$yA \leq w$ $\min w$.

These are dual linear programs.

Exercise 2: Show that they are indeed dual programs. Note that they are z and w are variables so neither are in standard or canonical form.

Question: Given an optimum of this "adversarial" LP? Does player have incentive to deviate?

No. He is only playing best response. Is positive only on responses whose values are equal to the value of the program.

11.2 Primal Dual algorithms

Question: Maximum weight perfect matching in a bipartite graph as a liner program?

$$\max_e w_e x_e$$

$$\forall u, \sum_{e=(u,v)} x_e = 1.$$

Question: Its dual program?

$$\min_e p_i$$

$$\forall e p_i + p_j \geq w_e.$$

Question: Feasible dual solution?

Set $p_i = 0$ for left nodes, and $p_j = \max w_e$ for the right nodes.

Question: Tight constraints?

Edges that achieve c_e.

Question: What should primal do?

Find matching, only using "tight" edges. (Complementary slackness says that if feasible dual and obey complementary slackness the value is optimal.)

Question: How?

Start at unmatched node on left, and follow alternating path, of tight edge and/or matched node on right until you find unmatched node on right.

Question: If not?

Find a set of nodes S where edges out of S are not tight.

Question: What to do?

Make them tight, drop p_i on all "left" nodes in the set and raise them on the right, until new edge is tight, i.e., drop p_i by the minimum over all edges $e = (i, j)$ out of S of $p_i + p_j - w_e$. This will make an edge tight and keep the dual feasible.

Exercise 3: Show that all edges in the current matching remain "tight".

Question: What have we done?

This process will augment the size of the matching by one. It maintains complementary slackness and dual feasibility.

Question: How long do we go?

Well, n augmentations.

Question: Optimal when done?

Sure. Obeys complementary slackness, and both are feasible.

Question: How much time per augmentation?

$O(n)$ dual adjustments per augmentation. $O(m)$ time per adjustment. $O(nm)$ time overall. With a bit of care, $O(n^2)$ time.

Exercise 4: How many times does one look at an edge in one augmentation? Show that this gives $O(n^2 + m)$ time per augmentation.

This is the hungarian method for maximum weight matching and represents an early (perhaps first) of many primal dual algorithms.