1 Spectral gap, expansion, and the co-area formula

Let $G = (V, E)$ be a graph with maximum degree $d_{\max}$, and let L be the Laplacian of G. Recall that for $x \in \mathbb{R}^n$, we have $x^T Lx = \sum_{ij \in E}(x_i - x_j)^2$, and

$$\lambda_2 = \min_{x \in \mathbb{R}^n : \sum x_i = 0} \frac{x^T Lx}{\|x\|^2}$$

Also recall that $\phi(S) = \frac{e(S,\bar{S})}{\min(|S|,|\bar{S}|)}$, and $\phi_G = \min_{S \subseteq V} \phi(S)$.

Let's define $B(x) = \sum_{ij \in E} |x_i^2 - x_j^2|$. We can use Cauchy-Schwarz to bound

$$\begin{aligned} B(x) = \sum_{ij \in E} |x_i^2 - x_j^2| = \sum_{ij \in E} |x_i - x_j|(x_i + x_j) &\leq \sqrt{\sum_{ij \in E}(x_i - x_j)^2}\sqrt{\sum_{ij \in E}(x_i + x_j)^2} \\ &= \sqrt{x^T Lx}\sqrt{\sum_{ij \in E}(x_i + x_j)^2} \end{aligned}$$

Now we bound the latter term by

$$\sqrt{\sum_{ij \in E}(x_i + x_j)^2} \leq \sqrt{2\sum_{ij \in E}(x_i^2 + x_j^2)} \leq \sqrt{2d_{\max}\sum_{i \in V} x_i^2} = \sqrt{2d_{\max}}\|x\|.$$

So we conclude that

$$B(x) \leq \sqrt{2d_{\max}}\|x\|\sqrt{x^T Lx}. \tag{1}$$

Now, let $y \in \mathbb{R}^n$ be an eigenvector with eigenvalue λ_2, and define $x \in \mathbb{R}^n$ by $x_i = 0$ if $y_i < 0$ and $x_i = y_i$ otherwise (i.e. x is the positive part of y). Let $V_+ = \{i \in V : x_i > 0\}$. Without loss of generality, $|V_+| \leq n/2$ (otherwise, we could have started with $-y$ which is also an eigenvector with eigenvalue λ_2).

Let $V_r = \{i \in V : 0 < x_i \leq r\}$, and note that for $r > 0$, we have $|V_r| \leq n/2$ by the preceding paragraph. It follows that $e(V_r, \bar{V}_r) \geq \phi_G |V_r|$. Now are going to use something called the "co-area formula" in geometry, which is nearly trivial, but important to understand:

$$B(x) = \int_{r=0}^{\infty} e(V_r, \bar{V}_r)\, d(r^2) \geq \phi_G \int_{r=0}^{\infty} |V_r|\, d(r^2) = \phi_G \sum_{i=1}^{n} x_i^2 = \phi_G \|x\|^2. \tag{2}$$

Using (1) and (2), we conclude that $\frac{x^T Lx}{\|x\|^2} \geq \frac{\phi_G^2}{2d_{\max}}$. So it only remains to show that $\lambda_2 \geq \frac{x^T Lx}{\|x\|^2}$, which we do as follows. For $i \in V_+$, we have

$$(Lx)_i = \deg(i)x_i - \sum_{j\,:\,ij \in E} x_j \leq \deg(i)y_i - \sum_{j\,:\,ij \in E} y_j = (Ly)_i = \lambda_2 y_i.$$

Using this, we can finish by writing

$$x^T Lx = \sum_{i \in V} x_i (Lx)_i \leq \lambda_2 \sum_{i \in V_+} y_i^2 = \lambda_2 \sum_{i \in V} x_i^2 = \lambda_2 \|x\|^2. \tag{3}$$