
Lecture 10

1 Cheeger's inequality

In the last lecture we introduced the notion of edge expansion, eigenvalues of the adjacency matrix and the averaging interpretation of the action of the normalized adjacency matrix M and stated Cheeger's inequality that relates the spectral gap to the expansion.

$$\frac{1 - \lambda_2}{2} \leq h(G) \leq \sqrt{2(1 - \lambda_2)} \quad (1)$$

Today we will prove the left side of Cheeger's inequality, the proof of the right side of the inequality is harder and we will see it in a future lecture.

Why is the left side of Cheeger easier? The left side of Cheeger's inequality is equivalent to proving that $\lambda_2 \geq 1 - 2h(G)$. It is easy to prove an inequality of the form $\lambda_2 \geq c$ using the Rayleigh quotient characterization of the second eigenvalue from the previous lecture,

$$\lambda_2 = \max_{x \perp \vec{1}} \frac{x^T M x}{x^T x} \quad (2)$$

In order to prove that $\lambda_2 \geq c$ it suffices to find a vector $v \in \mathbb{R}^n, v \perp \vec{1}$ such that the Rayleigh quotient $\frac{v^T M v}{v^T v} \geq c$. The averaging interpretation of the action of M is useful for bounding the Rayleigh quotient.

Proof idea: Given a partition (S, \bar{S}) of the vertices of G with edge expansion $h(S)$ the proof idea is to find vector $v \in \mathbb{R}^n, v \perp \vec{1}$ with Rayleigh quotient at least $1 - 2h(S)$. Applying the argument to the sparsest cut in G yields the left side of Cheeger's inequality,

$$\lambda_2 \geq 1 - 2h(G) \quad (3)$$

CLAIM 1

Given a partition (S, \bar{S}) of the vertices of G with $|S| \leq n/2$, define vector v such that $v_i = -|\bar{S}|$ for $i \in S$ and $v_i = |S|$ for $i \in \bar{S}$.

$$\frac{v^T M v}{v^T v} \geq 1 - 2h(S)$$

PROOF: The vector $v \perp \vec{1}$ by design as the vertices in S contribute $-|S||\bar{S}|$ to $\sum v_i$ which is cancelled by the $|S||\bar{S}|$ contributed by vertices in \bar{S} . In order to bound the Rayleigh quotient, we compute the quantities $v^T v$ and $v^T M v$,

$$v^T v = \sum_{i \in S} |\bar{S}|^2 + \sum_{i \in \bar{S}} |S|^2 = |S| \cdot |\bar{S}| \cdot (|S| + |\bar{S}|) = n|S| \cdot |\bar{S}| \quad (4)$$

If there are no edges in G crossing the partition (S, \bar{S}) then $Mv = v$ and $v^T Mv = v^T v$ by the averaging interpretation of the action of M . Consider the effect of adding an edge (i, j) across the partition. One of the terms in the average $\frac{1}{d} \sum_{k \sim i} v_k$ changes from $-\bar{S}$ to $|S|$, this results in a net increase of $\frac{|S| + \bar{S}}{d} = \frac{n}{d}$ in the average value. Arguing similarly, we find that the average value $\frac{1}{d} \sum_{k \sim j} v_k$ decreases by $\frac{n}{d}$.

Adding an edge (i, j) across the partition changes $Mv_i \rightarrow Mv_i + \frac{n}{d}$ and $Mv_j \rightarrow Mv_j - \frac{n}{d}$. The inner product between v and Mv changes by $-(|\bar{S}| + |S|)\frac{n}{d} = -\frac{n^2}{d}$ for the addition of every edge across (S, \bar{S}) . Therefore,

$$\begin{aligned} v^T Mv &= v^T v - \frac{n^2}{d} |E(S, \bar{S})| \\ &= n|S||\bar{S}| - n^2 |S| h(S) \end{aligned} \quad (5)$$

The equality $|E(S, \bar{S})| = d|S|h(S)$ follows from the definition of edge expansion. The value of the Rayleigh quotient is,

$$\frac{v^T Mv}{v^T v} = \frac{n|S||\bar{S}| - n^2 |S| h(S)}{n|S||\bar{S}|} = 1 - \frac{n}{|\bar{S}|} h(S) \geq 1 - 2h(S) \quad (6)$$

□

1.1 The spectral gap as a relaxation of conductance

Another perspective on Cheeger's inequality is the observation that the spectral gap $(1 - \lambda_2)$ is a relaxation of the optimization problem of computing the conductance $\phi(G)$. The spectral gap can be written in terms of the Rayleigh quotient,

$$\begin{aligned} 1 - \lambda_2 &= \min_{x \perp 1} \frac{x^T x - x^T Mx}{x^T x} = \min_{x \perp 1} \frac{d \sum_i x_i^2 - \sum_{ij} 2A_{ij} x_i x_j}{d \sum_i x_i^2} \\ &= \min_{x \perp 1} \frac{\sum_{ij} A_{ij} (x_i - x_j)^2}{d \sum_i x_i^2} \end{aligned} \quad (7)$$

The sum of the entries of x is equal to 0 as $x \perp 1$ so we have $(\sum x_i)^2 = \sum x_i^2 + \sum 2x_i x_j = 0$. The expression in the denominator of the above expression can be rearranged to obtain $d \sum_i x_i^2 = \frac{d}{n} \sum_{i,j} (x_i - x_j)^2$,

$$1 - \lambda_2 = \min_{x \perp 1} \frac{n \sum_{ij} A_{ij} (x_i - x_j)^2}{d \sum_{ij} (x_i - x_j)^2} \quad (8)$$

The expression for the spectral gap is invariant under shifting all the coordinates of x by a constant, so the constraint $x \perp 1$ can be changed to $x \in \mathbb{R}^n \setminus 0$. If x is restricted to the characteristic vector of a cut $\{0, 1\}^n \setminus 0$ the value of the expression (8) is the conductance of the cut defined by x . The conductance $\phi(G)$ can therefore be viewed as a relaxation of the spectral gap,

$$\phi(G) = \min_{S \subset [n]} \frac{nE(S, \bar{S})}{d|S||\bar{S}|} = \min_{x \in \{0, 1\}^n \setminus 0} \frac{n \sum_{ij} A_{ij} (x_i - x_j)^2}{d \sum_{ij} (x_i - x_j)^2} \quad (9)$$

The conductance is obtained by minimizing the expression (8) over characteristic vectors of cuts in $\{0, 1\}^n \setminus 0$ while the spectral gap is obtained by minimizing the same expression over $\mathbb{R}^n \setminus 0$. It follows that $\phi(G) \geq 1 - \lambda_2$ and using $2h(G) \geq \phi(G)$ we have another proof of the left side of Cheeger's inequality.