Lecture 6 and 7

Today we show that a large class of graphs, beyond just grids, have good partitions. For example, any graph that can be drawn in the plane essentially looks like a grid in terms of how easy it is to partition as specified by the following theorem.

Theorem 1
(Fredrickson, Lipton/Tarjan) There is a partition of any N-node planar graph into $\{S_0, \ldots, S_p\}$ for any ϵ, there is a c such that

1. $|E_i| \leq (1+\epsilon)|E|/p$ *and*
2. $|N(S_i)| \leq c\sqrt{N/p}$.

Here, inefficiency comes both from the imbalance in computation as defined by the ϵ as well as the cost of communicating the values of the neighboring nodes.

Using this theorem, one can again see that one can get an "efficient" algorithm for planar graphs when $N >> Lp$ and when $N = \Omega(g^2)$.

1 Finding good partitions.

We first see how to find a balanced separator. That is, given a graph $G = (V, E)$, find a subset of nodes C whose removal leaves A and B, such that $|A| \leq 2n/3$ and $|B| \leq 2n/3$, and there is no edge in E between a node in A and a node in B.

We start with an easy one.

Theorem 2
A tree has a separator of size 1.

Proof:

Direct each edge toward the side with more nodes. Remove the node with no outgoing arcs. Each component has size at most $n/2$.

Question 1. Prove the following claim.

Claim 3
Given a set of components $A_1, \ldots, A_k$, all of which have size less than $n/2$, one can group them into sets A and B each of which has size at most $2n/3$.

□

Now, we will try to prove the following interesting theorem.

Theorem 4
Any n-node node planar graph has a separator of size $3\sqrt{n} + 1$.

PROOF:

We will first prove the following lemma.

LEMMA 5
Any planar graph with a spanning tree of depth r has a set of nodes of size $2r+1$ whose removal leaves no connected component that is larger than $2n/3$.

We use the following fact.

FACT 1 Any closed curve in the plane divides the plane into two pieces, the inside and outside.

For us, we use this theorem by noting that the a planar graph can be drawn in the plane and a cycle in the graph corresponds to a closed curve in the plane. Thus, the nodes on the inside of the closed curve that corresponds to this cycle's embedding are separated from the nodes on its outside.

The first step is to triangulate each face in the planar embedding. A face is any region on the plane with nothing in it, it is bordered by a cycle in the planar graph.

Now, recall that if we have a spanning tree, that any other edge in the graph induces a cycle in the graph. That is, any of the edges in the triangulated graph that is not in the spanning tree creates a cycle in the spanning tree and separates the graph. We will argue that one of these cuts is balanced.

We consider a "dual" graph that we obtain by placing nodes in each face of the triangulated planar embedding and drawing edges between adjacent faces across the non-spanning tree edges. This forms a 'dual' tree, since a cycle in this dual graph would cut the plane (and the graph) into two pieces, which can't happen since the original primal tree spans the entire graph and the dual tree does not cross the primal tree. Moreover, the maximum degree of any node in the dual tree is at most 3 since each face is a triangle.

Again edge in the dual tree corresponds to an non-spanning tree edge (the edge between the faces that the dual edge "connects") in the original graph, which induces a cycle. Recall that each non-tree edge induces a cycle which correspond to a cut in the planar graph.

We point the dual edge towards the side of the cycle that contains more nodes. (Here, we consider the nodes on the cycle to arbitrarily belong to the outside of the cycle where outside is defined with respect to one exterior face.)

Here, we will find a node in the dual tree with only incoming edges. For this node, none of the associated cycles contain more than $n/2$ nodes. Moreover, for at least one of the incoming edges corresponds to a cycle whose interior and border contains at least $1/3$ of the nodes since the union of the cycles contains the whole graph. Thus, removing this cycle leaves only pieces of size $\leq 2/3$ of the whole graph. This proves the lemma.

Now, we use a very basic fact about all graphs.

FACT 2 Any level in a breadth first search tree of a graph separates all the nodes above the level from those below the level.

Thus, we first find a breadth first tree, and consider the levels $V_0, V_1, \ldots, V_d$. If $d \leq \sqrt{n}$, we can use the lemma to get the theorem we are seeking.

Otherwise, we can remove a set of levels $V_o, V_{o+\sqrt{n}}, \ldots, V_{o+i\sqrt{n}}$ where $o \leq \sqrt{n}$ such that the total number of nodes in these levels is at most $\sqrt{n}$. Essentially we are cutting every $\sqrt{n}$ level. So, we cut $1/\sqrt{n}$ of the nodes on average. We shift the starting point(o) to ensure that the we attain the average.

This separates all the nodes into pieces, where any piece can be augmented by adding one node to have a spanning tree of depth $\sqrt{n}$; Remove all the nodes in lower levels will leave an empty face, add a node that is connected to all nodes in the top level of the piece. Since every node in the piece can be reached in $\sqrt{n} - 1$ steps, the total depth is $\sqrt{n}$.

Only one piece can have size larger than $2n/3$. By using the lemma above, we can separate this piece into pieces of size smaller than $2n/3$ using $2\sqrt{n} + 1$ nodes. Using the results of Question 1 allows us to piece together the pieces to obtain a separator.

$\square$

Question 2: Show how to get a 50-50 cut, where the separator size is $O(\sqrt{n})$ in any planar graph. A 50-50 cut, is a partition A, B, C where $|A|, |B| \leq n/2$ and C is the separator. (Hint: Use the planar separator theorem above inductively.)

Question 3: Show that for any n, there is a graph with average degree at most 10 and maximum degree $O(\log n)$, where every balanced cut has size $\Omega(n)$? (Hint: let each node choose 5 random nodes to connect to, argue that any balanced cut has an very very (exponentially) low probability of being small, and use the union bound over balanced cuts.)