

# Unknots With Highly Knotted Control Polygons

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## Abstract

An example is presented of a cubic Bézier curve that is the unknot (a knot with no crossings), but whose control polygon is knotted. It is also shown that there is no upper bound on the number of crossings in the control polygon for an unknotted Bézier curve. These examples complement known upper bounds on the number of subdivisions sufficient for a control polygon to be ambient isotopic to its Bézier curve.

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There can be substantial topological differences between a curve and its control polygon, as depicted in Figure 1, which has control polygon  $P_0, P_1, \dots, P_5, P_0$ .

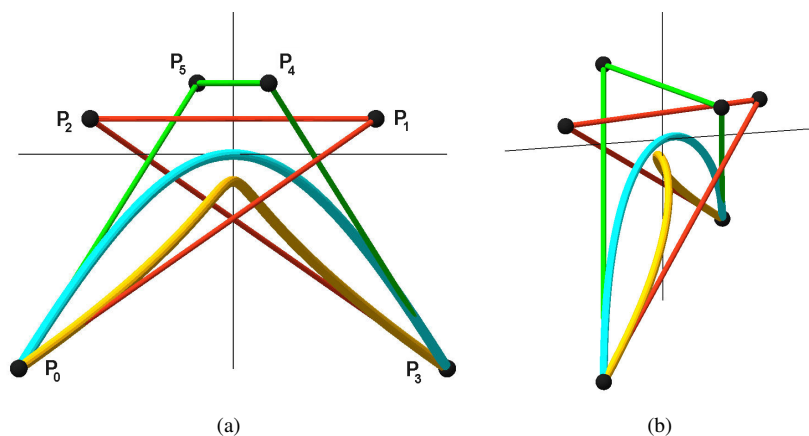


Figure 1: (a) Unknot with Knotted Control Polygon

(b) Rotated View

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<sup>1</sup>This author gratefully acknowledges partial support from NSF grant CCF 0429477 and an IBM Faculty Award. All statements here are the responsibility of the author, not of these funding sources.

A *knot* will be considered to be a closed, non-self-intersecting curve with a specific embedding in  $\mathbb{R}^3$ . When such a knot is described as a composite Bézier curve<sup>2</sup> for analysis or detailed geometrical manipulation, it would be highly advantageous if most of the calculations and operations could be done with just the linear segments of the control polygon. One topological difference is that the control polygon can have self-intersections when the associated curve does not [2]. However, as the control polygon of any Bézier curve is subdivided, it converges to the Bézier curve.

Any non-self-intersecting  $C^1$  composite Bézier curve with regular parameterisation will, *after sufficiently many subdivisions*, have a non-self-intersecting control polygon [3], so that the curve and its control polygon are homeomorphic<sup>3</sup>. The options for embedding closed curves in the plane are quite restricted so that homeomorphism is the crucial topological equivalence relation between closed planar curves, with the salient distinction being the presence or absence of self-intersections. However, there are significant differences in how closed, non-self-intersecting curves can be embedded within  $\mathbb{R}^3$ , as formally captured by ambient isotopy (Definition 0.1.) Ambient isotopy is a fundamental concept in knot theory [4]. Practical applications of ambient isotopy appear in geometric modeling, visualization and animation [5].

**Definition 0.1.** A continuous function  $H : \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}^3$  is an **ambient isotopy** between subsets  $X$  and  $Y$  of  $\mathbb{R}^3$  if  $H(\cdot, 0)$  is the identity,  $H(X, 1) = Y$ , and for each  $t \in [0, 1]$ ,  $H(\cdot, t)$  is a homeomorphism from  $\mathbb{R}^3$  onto  $\mathbb{R}^3$ . The sets  $X$  and  $Y$  are then said to be **ambient isotopic**.

Within  $\mathbb{R}^3$ , there exists a rich class of non-self-intersecting, composite, cubic Bézier curves for which each curve will, *after sufficiently many subdivisions*, be ambient isotopic with its control polygon [6]. That work gave no specific examples where the control polygon and curve were in different isotopy classes. That upper bound applies to Example 0.1, where ambient isotopy is achieved after only one subdivision<sup>4</sup>.

**Example 0.1.** Denote by  $\mathbf{c}$  the closed, composite cubic Bézier curve with control points,  $P_0, P_1, \dots, P_5, P_0$ , respectively listed as:

$$(-6, -6, 12), (4, 1, -1), (-4, 1, 1), (6, -6, -12), (1, 2, 4), (-1, 2, -4).$$

**Proposition 0.1.** Curve  $\mathbf{c}$  is the unknot but the control polygon of  $\mathbf{c}$  is a trefoil<sup>5</sup>.

**Proof:** Let  $K$  denote the control polygon of  $\mathbf{c}$ . This is a closed, non-self-intersecting curve embedded in  $\mathbb{R}^3$ . Note that  $K$  has two superfluous undercrossings:

$$[P_2, P_3] \text{ under } [P_5, P_0] \text{ and } [P_2, P_3] \text{ under } [P_0, P_1].$$

<sup>2</sup>For composite Bézier curves, the word ‘knot’ is also used for a ‘junction point’ [1].

<sup>3</sup>A continuous function  $f : X \rightarrow Y$  is a **homeomorphism** between  $X$  and  $Y$  if  $f$  is bi-continuous, 1-1 and onto. The sets  $X$  and  $Y$  are then said to be **homeomorphic**.

<sup>4</sup>Smooth curve examples are presented elsewhere [7].

<sup>5</sup>A trefoil is the knot with three crossings.

The remaining three crossings are alternating, as can be verified by linear interpolation, so that a Reidemeister move [8] on  $K$  permits the conclusion that  $K$  is ambient isotopic to a piecewise linear trefoil.

The vertices of  $K$  are the control points for a composite cubic Bézier curve,  $c$ . To show that  $c$  is the unknot, it suffices to show that there are no self-intersections in the projection of  $c$  into the plane  $z = 0$ . Denote by  $\tilde{c}_1$  the component created from projection of the control points  $P_0, P_1, P_2, P_3$  and by  $\tilde{c}_2$  the component created from projection of the control points  $P_3, P_4, P_5, P_0$ . Both  $\tilde{c}_1$  and  $\tilde{c}_2$  are assumed to be parameterized over  $[0, 1]$ . Let  $d_2$  denote the Euclidean distance function in  $\mathbb{R}^2$ .

The curve  $\tilde{c}_1$  is a variant of a published example [2, Figure 1.11(e)], and it follows from that example that  $\tilde{c}_1$  is non-self-intersecting because  $d_2(\tilde{P}_1, \tilde{P}_2) < d_2(\tilde{P}_0, \tilde{P}_3)$ , where these distance arguments are the projections of  $P_1, P_2, P_0, P_3$ , respectively. The curve  $\tilde{c}_2$  is non-self-intersecting because of its convex control polygon. It remains to show that these components intersect only at their end points.

The subdivided control polygons in the half-plane  $x \leq 0$  is shown with solid black lines in Figure 2. Subdividing  $\tilde{c}_1$  and  $\tilde{c}_2$  at  $t = 0.5$  gives control points in the half-plane defined by  $x \leq 0$  of

$$\begin{aligned} \tilde{c}_1: & (-6, -6), (-1, -2.5), \\ & (-0.5, -0.75), (0, -0.75); \\ \tilde{c}_2: & (0, 0), (-1.75, 0), \\ & (-3.5, -2), (-6, -6). \end{aligned}$$

These convex hulls within the half-plane  $x \leq 0$  intersect only at  $P_0$ , shown in Figure 2.

The proof is completed by using the symmetry about the  $y$ -axis. ■

**Highly Knotted Control Polygons:** We can now use this configuration to form an unknot with a control polygon of arbitrary knottedness.

Denote four instances of  $K$  knot of Figure 1 as  $K_1, K_2, K_3, K_4$ . For  $i = 1, \dots, 4$ , cut open each  $K_i$  at its initial point and then join the  $K_i$  in the pinwheel pattern shown in Figure 3. Two new knots are formed; a polygonal knot with 12 crossings and a composite Bézier curve of the unknot. This process can be formalized as a connected sum of knots [4] and generalized for each  $n \geq 1$  to exhibit a knotted control polygon with  $3n$  crossings for a Bézier unknot. This lack of an upper bound on the number of crossings in the control polygon leads to radically different embeddings in  $\mathbb{R}^3$  compared to the corresponding Bézier unknot. This construction for Figure 3 is very specialized and relies upon the convex hull of consecutive control polygons being disjoint except at the common vertex and also that non-consecutive convex hulls are pairwise disjoint. As  $n$  increases, it becomes appropriate to scale the control polygons to meet these criteria.

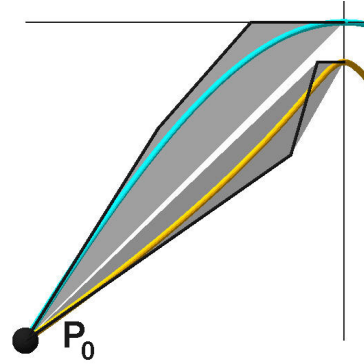


Figure 2: Subdivision & Convex Hulls

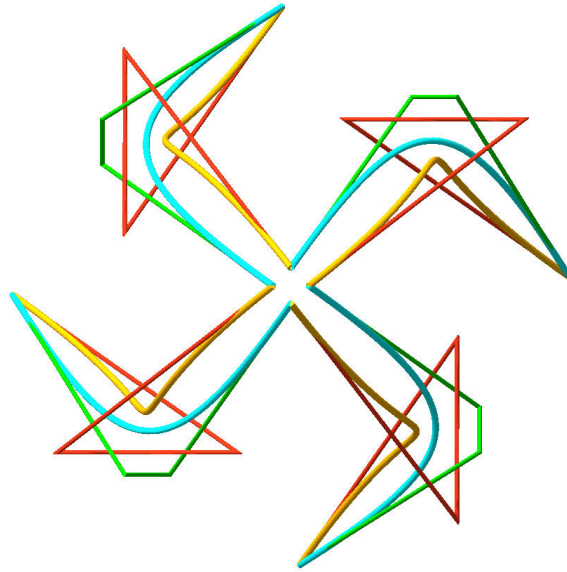


Figure 3: Connected Sum for the Control Polygon of a Bézier Unknot

The authors thank the reviewers for their constructive criticism.

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