Interlinking Polyhedral Wire-Frames

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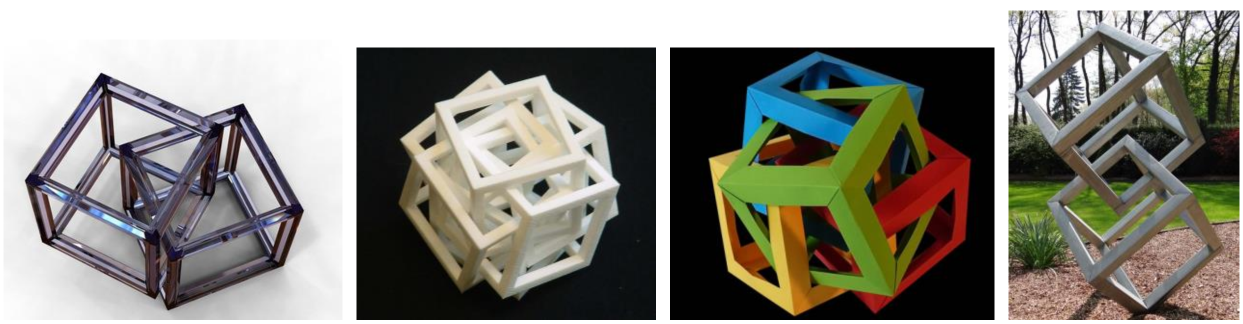
Abstract

This paper explores in how many topologically different ways the wire-frames of two cubes or two tetrahedra can be interlinked. Open questions remain. Some select configurations are proposed as abstract sculptures.

1. Introduction

Several artists have created sculptures comprising interlinked edge-frames of cubes. Pairs of interlinked cube-frames appear in virtual form on the homepage of *Cross Collaborate* [2] (Fig.1a) and in physical reality in a sculpture by Lammers [8] (Fig.1d). Multiple interlinked cube-frames have been fabricated on a 3D printer by Roelofs [10] (Fig.1b) and constructed out of origami pieces by Eisner [3] (Fig.1c).

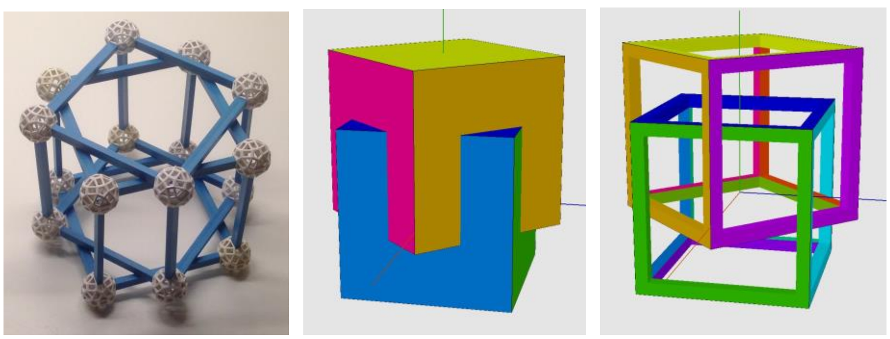


(a) (b) (c) (d)

Figure 1: *Cube frame sculptures: (a) 'Intersecting Crystal Cubes' [2], (b) '6 Cubes' by Roelofs [10], (c) 'Intersecting Cubes' by Eisner [3], (d) 'Cubes In Each Other' by Lammers [8].*

There exist an infinite number of ways that such frames can be placed into interlinked configurations. In this paper we investigate in how many topologically different ways just two identical frames can be interlinked. This question was raised in April 2015 by the first author (Walt) in an email to the second author (Carlo) and was accompanied by pictures of seven cube-frame models made with ZomeTool [12] sticks and balls (Fig.2a). Intrigued by this problem, Carlo soon found additional linkings that seemed to be different (Fig.3a). In several more e-mail exchanges over the subsequent week the number of different models found doubled, and the authors started to speculate whether there might be more than 20 different linkings of two cube-frames. The problem seemed rather tricky, since it was not immediately clear, when two linkages should be considered truly "different." A purely topological definition in which the wire frames are considered totally flexible graphs makes no sense. Pairs of edges could be twisted around each other in complicated ways and the resulting shapes might not look like cubes at all. Considering the wire frames to be totally rigid, and allowing only movements that are possible between such rigid frames causes a different set of problems. The test of sameness could not be reduced onto an analysis of the linkings between different face loops, but would have to involve a simulation of all physically possible movements. Thus we settled onto the following definition:

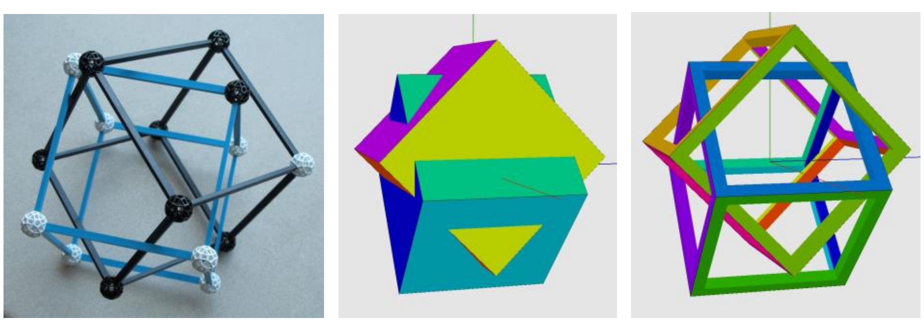
We consider the cube-frames to be made from flexible wires that can be bent out of shape by a fair amount, but which then snap back to the perfect geometrical shape of a regular cube. The legal configurations that we check for "same-ness" would always have the wire-frame assume the perfect polyhedral geometry and have some spatial clearance around all wire-edges – which may be arbitrarily small. All configurations that can be obtained manipulating the frames while using this flexibility – without turning them inside-out (which would change their orientation), and without passing any edges through one another, "are the same." By this definition, the two mirror images of a chiral configuration will be called "different." While this definition makes the problem unambiguous, it still leaves open questions: How do we assure that two configurations that may look quite different cannot be transformed into one another by some tricky maneuver? How do we find all possible different configurations and make sure that indeed we have accounted for all of them? As the universe of discovered cube-frame linkings grew more and more complex, we started a parallel investigation of interlinked tetrahedral frames – for which we hoped to arrive more quickly at a complete enumeration of all possibilities.



	M	Z	R	O	Y	L
G	⊖			⊕		
A		⊖	⊕			
C	⊕		⊖			
U		⊕		⊖		
B						
P						

(a) (b) (c) (d)

Figure 2: *(a) Two interlinked cube-frames with D_{4d} symmetry, ("4P4P"), (b) the same configuration with solid cubes, (c) virtual cube frames, rotated 90°, (d) corresponding link matrix.*



	M	Z	R	O	Y	L
G						
A						
C						
U						
B					⊖	⊕
P					⊕	⊖

(a) (b) (c) (d)

Figure 3: *(a) Two differently linked cube-frames with D_{2d} symmetry ("2O2O"), (b) the same configuration with solid cubes, (c) virtual cube frames rotated 180°, (d) corresponding link matrix.*

2. Link Matrices

Matthias Goerner, a former math student at UC Berkeley, was also drawn into the discussion, and he encouraged us to use a matrix showing all the pairwise linkages of all the face-loops in the two wire frames to find out when two configurations are different. First we "personify" the various face-loops by assigning each face in both cubes a different color. We use the following twelve colors: **M**agenta, **Z**innober, **R**ed, **O**range, **Y**ellow, **L**ime in one cube, and **G**reen, **A**qua, **C**yan, **U**niform, **B**lue, **P**urple in the other one (Fig.2b,c; 3b,c). Now we write out a 6×6 matrix that explicitly specifies for each "bipartite" loop pair whether it is linked or not (Fig.2d, 3d). Linkings are signed using the following convention: Each face is assigned a CCW orientation as seen from the outside of the cube, and a face-normal vector that points outward. When a face-loop passes through another loop in the direction of the face normal, we assign this link a "+1", and in the opposite case, we label it as "−1". Each face-loop must have an even number of linkings with the other cube, and the "+1" and "−1" labels must alternate as the face-loop dives in and out of the other cube (perhaps repeatedly). Since the face loops are all planar, two loops cannot have a linking number larger than 1. Thus we just enter "**+**" or "**–**" signs into the link matrix.

This link matrix is definitely an important discriminator. All cube-frame manipulations that do not involve any edge-crossings cannot change the pairwise linking of any two face-loops; thus all such configurations must have the same link matrix. But the question remains, whether two configurations that cannot be transformed into one another might still have the same link matrix. Moreover, our cube frames under investigation are not colored. Thus any symmetry-operation applied to one of the cubes will not result in a new configuration, but nevertheless will produce a different link matrix. Thus for each configuration we must generate all $24 \times 24 \times 2 = 1152$ link matrices resulting from the orientation-preserving symmetry transformations on each cube (24), as well as the operation of swapping the roles of the two cubes. This matrix-set as a whole will then be used to characterize a specific configuration. Since it would be unwieldy to compare such large sets against one another, we represent each set with a "canonical" matrix. This matrix is obtained by assembling the six rows of each matrix into a 36-component record and sorting these records in ascending order. The "minimal" record, with the most consecutive "−1" in the first few fields, is chosen as the canonical representative.

The link matrices generated when all combinations of symmetry transformations are applied to the two cubes need not all be different. In Figures 2 or 3 both cubes can be rotated around their joint (vertical) symmetry axis by the same angle without changing their relative positioning or the entries in the link matrix. The configuration in Figure 2, in which the four interlinked faces form a prism (P) in each cube, exhibits 8-fold D_{4d}-symmetry, and every matrix type will show up 8 times when all 1152 combinations of symmetry operations are applied; thus there will be only 144 <u>different</u> matrices generated. Figure 3 exhibits 4-fold D_{2d}-symmetry, and the same matrices will appear at least 4 times. However, this configuration has other symmetry transformations on individual cubes that do not change the matrix either: Each cube individually can be rotated through 4 positions that do not change the interlinking: Think of the two frames as two interlinked facetted belts going through a pair of opposite (O) faces in the other cube. This reduces the number of different matrices generated by another factor of 16. So there will only be (1152/4)/16=18 different matrices. We can verify independently that 18 is the maximum number of different matrices that we can expect for this configuration: There are only three ways in which pairs of opposite faces can be selected on a cube, yielding nine different combinations of four fields in the matrix that have non-zero entries. For each such selection, there are only two legal options of placing the "+1"s and the "−1"s at opposite rectangle corners.

3. Computer Search for New Configurations

It is rather tedious and error-prone to construct physical models from ZomeTool and to tag the edges with colored tape to determine the link matrix for any potentially new configuration. So we soon started to create colored virtual models with suitable CAD tools. Carlo used Berkeley SLIDE [1], while Walt used the powerful combination of Grasshopper [6] and Rhinoceros [9]. Walt enhanced his program to automatically determine the corresponding link matrix, to calculate the complete matrix-set by applying all orientation-preserving symmetry transformations, to find the canonical link matrix representing this set, and to compare this one against all previously found canonical matrices. Later, the program was enhanced further to generate also the mirror image of a given configuration and to check whether this leads to the same canonical matrix, indicating that this configuration had mirror symmetry – or whether there existed a pair of topologically different, chiral configurations.

After the development of these programs, it became more convenient to do the experimentation in this virtual CAD space. Promising configurations were entered into the computer by specifying a rigid-body transformation for one of the two cube frames, while the other one was assumed to stay in a symmetrical position around the origin of the coordinate system. Interactive graphics allowed the user to gradually turn or shift one of the cubes until another edge-crossing occurred and to find out quickly whether this was a novel configuration. Subsequently, it took only one small additional step to automate the search completely. The computer repeatedly chooses 6 position parameters (3 Euler angles and translations in x, y, and z) for the movable cube at random, but in a range that promises to have some overlap between the two cubes, and then checks the novelty of the current configuration. Within a few weeks, this program increased the number of known cube-frame linkings from 30 in mid-June to 91 by mid-July, and to 144 by mid-October. The last discoveries came in at a rather slow rate, because some of those configurations have very tight margins: If one of the angles of rotation is changed by only 0.05 degrees, or the cube position is shifted by 0.052% of the cube edge length, the configuration turns into another, previously known one. In one such experiment we removed one of these hard-to find configurations from the known set; it then took the computer program 531'000 trials to rediscover this particular linking! Our last new solution found took 700'000 samples in this 6-dimensional space. So we believe that we have found <u>almost all</u> different configurations, – but we cannot be completely sure!

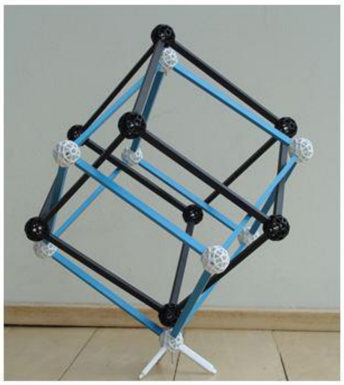
4. Preliminary Results for Linked Cube-Frames

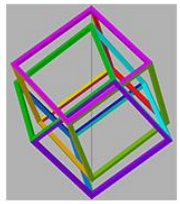
Page limits constrain us to presenting only a high-level summary of our results on interlinked cube-frames (Table 1) and to briefly describe a few special cases. Configurations are characterized by the number of linked edge-loops in each cube and by the maximal symmetry that the configuration may attain. Configurations in which the cubes play indistinguishable roles are called "swap-symmetric" and are denoted with an appended "s". If the role of the two cubes is different, in particular if they have a different number of their edge-loops being linked, a "d" is appended instead. Linkings that allow to be placed in a mirror symmetric configuration, are further characterized with an appended "m". If two mirror configurations exist that cannot be transformed into one another without any edge-crossing moves, they are called "chiral" – denoted with an appended "c". Each entry ending in a "c" in Table 1 reflects a set of half as many mirror pairs.

Table 1: *Number of Topologically Different Linkings of Two Cube-Frames.*

	2 edge-loops	3 edge-loops	4 edge-loops	5 edge-loops	6 edge-loops	
2 edge-loops	2sm, 1dm					3
3 edge-loops	0	1sm, 1sp, 2sc, 2dm				6
4 edge-loops	0	2dc	1sm, 10sc, 2dm, 10dc			25
5 edge-loops	0	0	1dm, 20dc	16sc, 3dm, 1dp, 24dc		65
6 edge-loops	0	0	3dm, 2dc	26dc,	14sc	45
					total:	144
	s: swap symmetry		m: mirror symmetry			
	d: different roles		c: chiral configuration		date:	1/20/2016
	for the two frames		p: pseudo chirality			

Determining what symmetries exist in a particular model, or whether two mirror configurations can be transformed into one another, is not an easy task. For the more complex linkings, we rely on the number of different link-matrices associated with such a configuration. If the number of link-matrices doubles when we swap the role of the two cubes, then we conclude that this configuration has no real swap symmetry. Similarly, we can only be sure that we have a chiral pair, if the application of a mirror transformation doubles the number of different link-matrices generated. There are a few cases where the transformation between two mirror images must pass through an illegal, over-constraint half-way-state with bent edges. We call such configurations “pseudo-chiral” and denote them with an appended “p”. An explicit example of this will be discussed in the section on tetrahedral frames. Here we just show one special case that links all six faces in both cubes in a highly regular manner; the configuration has swap symmetry and is chiral, but it also has overall 6-fold D_3 symmetry (Fig.4).





	M	Z	R	O	Y	L
G				⊖	⊕	
A			⊖			⊕
C	⊕					⊖
U		⊕			⊖	
B		⊖	⊕			
P	⊖			⊕		

Figure 4: *Chiral Cube Linking.*

The main lack in our findings is a conclusive way to assure that we have found <u>all</u> possible linkings – something more trustworthy than exhaustively sampling 6D parameter space! The method described in the next section offers some promise, but it seems too difficult to apply it manually to the more than a hundred cube-frame linkings. However, for the smaller scope of all tetrahedral linkings this seemed doable – so we decided to re-focus our investigation onto interlinked tetrahedral frames, for which we can present here a much more detailed analysis and a complete picture. More details about the cube frames will eventually be published in a forthcoming Technical Report.

5. Evolutionary Graph

When we move two polyhedral wire-frames against one another, so that just one pair of edges undergoes an “edge-cross” move, exactly four face loops will be affected and change their mutual linking state. Since they were not originally linked, they will now be linked. The first contact between two frames results in a “2-2” configuration. From this first, most loosely linked configuration, we may now explore what other “edge-cross” moves can be made that lead to more highly linked configurations. For a pair of tetrahedral frames it turns out that from the initial “2-2” configuration we can directly reach “3-3”, “2-4”, “3-4”, and “4-4” configurations.

Our goal is to build up gradually the complete graph that tells us what pairs of configurations are “neighbors” that can be transformed into one another via a single “edge-cross” move. If for every configuration we can account for all existing “exit” paths via all possible “edge-cross” moves, then we should obtain the complete “evolutionary graph” that leaves no place where any undiscovered configuration could hide. The advantage over the random sampling technique described in Section 3 is that this is a discrete search that does not depend on any sampling steps being fine-grained enough. On the other hand, it may be quite difficult to figure out what all the possible “edge-cross” moves are by which one might exit from the current configuration, because those possibilities are highly constrained by the actual geometry of the configuration. This problem is intricate enough that we have not yet found a way to cast it into a computer program.

6. Interlinked Tetrahedral Frames

As for the cube frames, our investigation started out with some physical models made with ZomeTool (Fig.5a). We also made models from pipe cleaners (Fig.5b); even though these models are less precise, they are easier to manipulate and to make incremental changes in the linking, where just one edge had to be opened to enable a desired edge-cross step. Also, most importantly, they allow to color differently the various edge loops, which then made it easier to manually fill in the link-matrix. Walt even made a model from springs and custom-made 3D-printed corners (Fig.5c) to obtain a model that had the flexibility as well as the stability and precision of the frames that were underlying our conceptual definition of when two configurations are the same. However, eventually the most important tool was still the program that Walt developed in Rhinoceros [9] and Grasshopper [6], which automatically calculates the link matrix (Fig.5e) for any CAD model entered (Fig.5d).

(a) (b) (c) (d) (e)

Figure 5: *Modeling tetrahedral linkings: (a) with Zometool, (b) with pipe-cleaners; (c) with springs and 3D-printed corners; (d) SLIDE [1] model with colored faces, (e) corresponding linking matrix.*

The tetrahedron has 12 orientation-preserving symmetries. Thus for any specific configuration found by using a pair of colored frames, we need to generate the matrices for 12×12×2=288 different color assignments and find the corresponding "canonical matrix" (now chosen from a 16-component record). We will discuss the different linking configurations in order of increasing complexity, as determined by the number of face loops involved in each linking, and in order of decreasing symmetry.

Picking 2, 3, or 4 edge loops on a tetrahedron can be done only in a single geometrical way, so the two numbers that specify how many loops in each tetrahedron are linked need not be characterized further as is useful in the case of the cube frames. In our "temporary" naming scheme (column 1 in Table 2), we have added an "A" for achiral configurations that can be transformed into a mirror-symmetric state. On the other hand, an "L" denotes one sibling of a truly chiral pair of configurations that exists in two mirror symmetric states with different link-matrix sets.

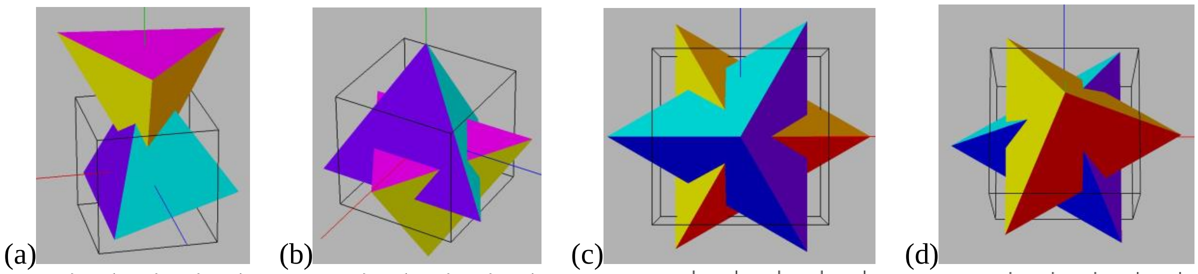
2-2-Linking

The simplest way to link two tetrahedral wire-frames is to let an edge of one frame cross an edge of the other one (Fig.5a,d). There is only a single type of such a 2-2-linking for tetrahedral frames. In a tetrahedron all ordered pairs of two faces are equivalent; they can be brought into coincidence with any other ordered pair through one of the 12 orientation-preserving symmetry-transformations ("OPST") on the tetrahedron. We use again a colored CAD model (Fig.5d) to explicitly identify the various interlinked face-loops and to write out the linking matrix (Fig.5e). If the right tetrahedron is flipped through 180° around the (red, horizontal) *x*-axis, the signs of the four entries in the linking matrix will be reversed – but the uncolored edge frame configuration still looks the same.

We now verify that we have accounted for all possible legal link-matrices for a 2-2 configuration: Each such matrix has exactly two non-empty rows and two non-empty columns. There are 6×6 ways to pick a combination of a certain row-pair and column-pair. Subsequently, there are two ways to place the required "+1" and "–1" entries at the corners of the rectangle formed by the two chosen rows and columns. This yields 72 possible legal matrices. This configuration can assume D_{2d}-symmetry and thus has four OPST. Applying those symmetry transformations to the whole configuration will not change the link matrix; thus every possible matrix must show up 4 times when all 288 OPST are applied (including the swapping of the two tetrahedra) – and 4×72 is indeed 288.

3-3-Linkings

The next more complicated linkings involve three loops in each tetrahedron. To make a simple, <u>symmetrical</u> 3-3-linking, we let two tetrahedra approach each other vertex-to-vertex, and then let the vertices just pass through each other (Fig.6a). Note that the magenta (M) and the green (G) "base" faces are <u>not</u> involved in any linking. Even if the two tetrahedra are pushed closer into one another, so that the two leading vertices punch through these base faces in the other tetrahedron, we still do not count this as a new configuration, as long as no additional edge-crossings have occurred. But if we continue this vertical movement, eventually the side edges in one frame will cross the base edges in the other frame and a new configuration results (Fig.6b). Again the question arises: How many different 3-3-linkings can there possibly be?



(a) (b) (c) (d)

(a)

	M	R	O	Y
G				
C			⊖	⊕
B		⊖	⊕	
P		⊕		⊖

(b)

	M	R	O	Y
G				
C			⊕	⊖
B		⊕	⊖	
P		⊖		⊕

(c)

	M	R	O	Y
G				
C			⊖	⊕
B		⊕		⊖
P		⊖	⊕	

(d)

	M	R	O	Y
G				
C			⊕	⊖
B		⊖		⊕
P		⊕	⊖	

Figure 6: *Four symmetrical 3-3-configurations of colored tetrahedra and their linking matrices.*

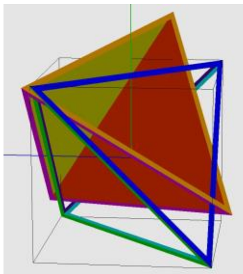
Four different-looking configurations are shown in Figure 6. However, if we ignore colors or swap the roles of the two colored tetrahedra, then 6(c) and 6(d) describe the same configuration. Again, let's determine the number of possible legal link matrices of type "3-3." There are 4×4=16 possibilities to select a pair of non-linked faces, which corresponds to an empty row/column combination in the matrix. If we ignore these empty row/columns, we are left with a 3×3 submatrix, which can be filled in only 2×2×3=12 possible legal ways: Identical symbols are either aligned in the direction of the main diagonal or along the minor diagonal; the "+1"s may lie (cyclically) above or below the "–1"s; and for each such choice there are 3 positional placements of these diagonal lines of identical symbols (corresponding to a rotation of one of the tetras around the shared 3-fold symmetry axis). This yields a total of 192 different legal link matrices.

Now we consider the symmetries of the possible 3-3 linkings. Configurations 6(a) and 6(b) both can exhibit D_{3d}-symmetry, with 6 OPST – which all generate the same link-matrix. These configurations account for 288/6=48 different matrices each. Figures 6c and 6d show configurations that have only C_{3v} symmetry and thus only 3 OPST; they can generate a maximum of 288/3=96 different link-matrices – but we only count this once. 48+48+96=192; which shows there are indeed only three different 3-3-linkings.

If we perform a mirroring operation on the upper tetrahedron in Figure 6a by switching the last two columns, turning (MROY) into (MRYO), then we obtain matrix 6(d) – which already belongs to a different configuration! So we see that allowing mirroring operations would render the linking matrix set an insufficient discriminator! Note also that the matrices for 6(a) and 6(b) are symmetric around the main diagonal; this indicates that these two configurations have swap-symmetry. On the other hand, swapping the two tetrahedra in Figures 6(c) and 6(d), which amounts to flipping the matrix around its main diagonal, transforms these two configurations into one another.

?-4-Linking

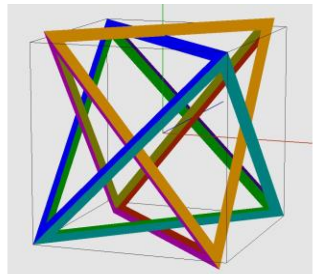
The next step is to involve all four edge-loops in one or in both of the tetrahedra. Now things get more complex and more intriguing. It is actually possible to have all four loops linked in one tetrahedron, while only two loops are linked in the other one; this requires that both those loops have to link with all four loops in the other tetrahedron. Figure 7a shows how such a 2-4-linking is possible; the solid yellow and red faces in the back are not linked at all, while the magenta and orange loops weave through all four edge-loops of the "stationary" tetrahedron (inside the grey cube-wire-frame). There is also a pair of chiral 3-4-linkings, where just one of the 3 loops is linked with all 4 loops in the other tetrahedron. This configuration is chiral and thus shows up as a pair of enantiomers. Clearly, all of these configurations cannot be swap-symmetric, since the number of linked loops differs in the two frames.



(a)

	M	R	O	Y
G	⊕		⊖	
C	⊕		⊖	
B	⊖		⊕	
P	⊖		⊕	

(b)



(c)

	M	R	O	Y
G	⊕	⊖		
C	⊖	⊕		
B			⊖	⊕
P			⊕	⊖

(d)

Figure 7: *Examples of ?-4-linkings: (a) 2-4A1 and (b) its matrix; (c) 4-4A1 and (d) its matrix.*

Finally there are five more 4-4-linkings. Their properties are shown in Table 2, which we believe to be a complete tabulation of all possible configurations.

Table 2: *All Topologically Different Configurations of Interlinked Tetrahedral Frames.*

Temp. Name	Sorted Links	Swap symm?	Mirror symm?	Maximal Symmetry Schönflies	Conway	# Matrices ind.	swap	refl.	Rotation rx	ry	rz	Translation tx	ty	tz	Max.Diam. % of edge
2-2A1	2200/2200	yes	yes	D2d	2*2	72	72	72	0	0	0	0	0	1.28	25.88%
3-3A1	2220/2220	yes	yes	D3d	2*3	48	48	48	90	0	0	0.73	0.73	-0.73	25.88%
3-3A2	2220/2220	yes	yes	D3d	2*3	48	48	48	90	0	0	-0.42	0.42	-0.42	14.94%
3-3A3	2220/2220	no	yes	C3v	*3	48	96	96	-69.1	44.2	58.7	0.34	-0.34	0.22	10.40%
2-4A1	4400/2222	no	no	C1	1	36	72	72	-12.6	4.5	12.1	0.23	0.16	0.24	1.67%
3-4C1	4220/2222	no	no	C1	1	144	288	576	43.1	-27.1	-3.6	0.29	0.31	-0.33	3.05%
4-4A1	2222/2222	no	pseudo	D2d	2*2	36	36	36	-25.9	34.4	11.8	-0.21	-0.18	-0.2	1.11%
4-4C1	2222/2222	yes	no	D2	222	72	72	144	0	0	114.7	0	0	0.97	7.16%
4-4C2	4222/4222	yes	no	C2	22	144	144	288	-30.7	30.9	4	0.05	-0.22	0.37	3.64%
4-4C3	4222/4222	yes	no	C1	1	144	144	288	-41.8	-40.8	25.6	-0.38	0.74	-0.37	2.05%
4-4C4	4422/4422	yes	no	D2	222	72	72	144	-0.02	10.9	0.12	0	-0.25	0	1.56%

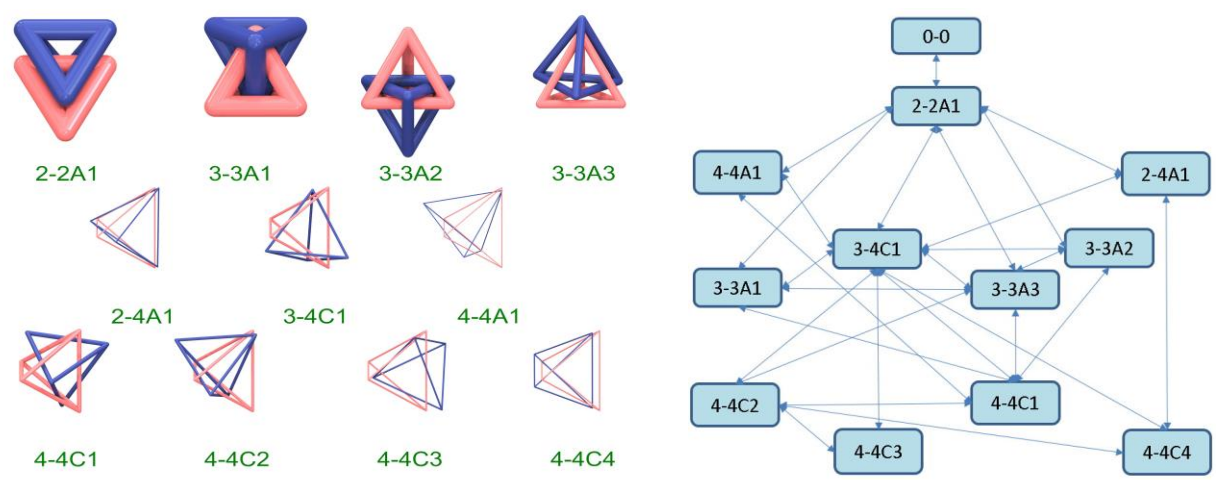
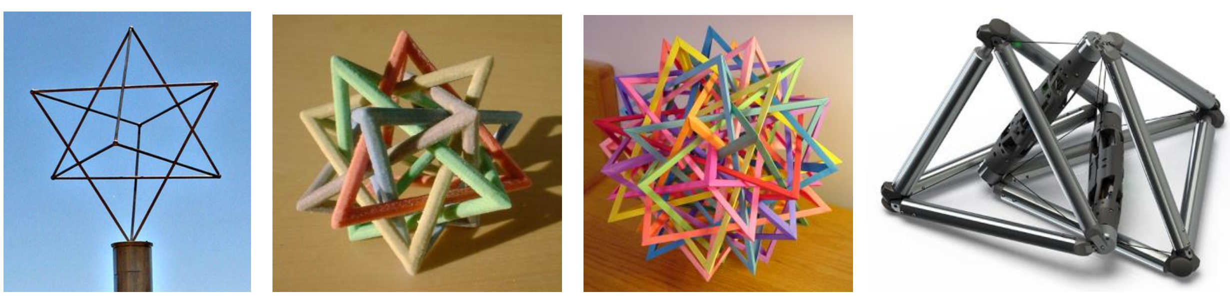


Figure 8: *All known tetrahedral linkings: (a) pictorial view, (b) evolutionary graph.*

Table 2 gives a (tentative) working name for each configuration, and lists with how many other loops each edge-loop is linked. It shows the kind of symmetries inherent in each configuration, the transformations that need to be applied to one tetrahedron to realize this configuration, and the largest diameter the edges may assume without intersecting with edges of the other frame. Next to the symmetry columns there are three columns that show the number of different link-matrices generated when all OPST first are applied to each tetrahedron individually, then with the inclusion of the swapping operation, and finally after mirroring the whole configuration. The red numbers still present some puzzles, since those numbers do not agree with the predictions derived from the actual symmetries observed on the physical model. Only one member of each chiral pair is listed in this table. Figure 8a gives a pictorial view of all eleven configurations; they are shown in their most relaxed position with maximally thick, non-intersecting edges. These local maxima were found with an evolutionary solver in Grasshopper that implemented a gradient descent algorithm. Figure 8b gives the evolutionary graph for this collection. An edge between two nodes implies that there is a single edge-cross move that transforms one node into the other. Figure 7c shows the interesting case of the pseudo-chiral configuration 4-4A1 in its illegal, symmetrical, intermediary position. To show clearly the nature of the topological linking, the MROY tetrahedron has been stretched vertically and shrunk non-uniformly in the x-y-directions.

7. Art Based on Linked Polyhedral Frames

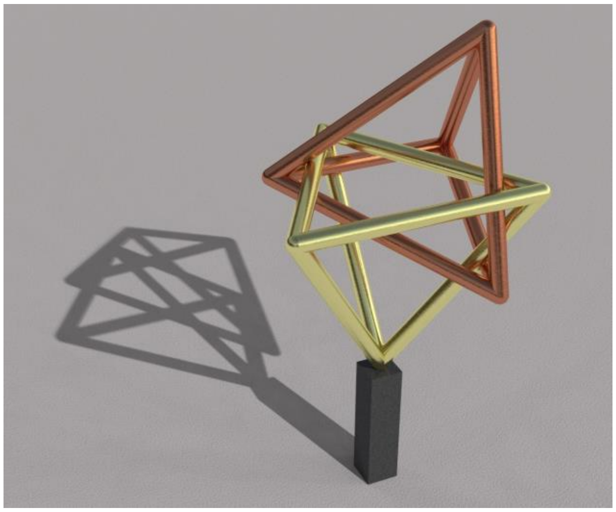
Sculptures depicting interlinked tetrahedral frames are harder to find than those depicting linked cubes. One example is *Star Tetrahedron* by Stephen Fitz-Gerald [4] (Fig.9a); it seems to correspond to configuration "3-3A2" (Fig.6b). The structure that most frequently shows up in a search for "interlinked tetrahedral frames" is the famous *Compound of 5 Tetrahedra* – often realized as a 3D-print [11], (Fig.9b) or as an origami construction. Origami constructions can also be found for other linkings of tetra-frames, ranging from two to 15 (Fig.9c). Other sculptures involving tetrahedral frames often have intersecting frame edges, or are jumbled compositions of many imprecisely mitered tetrahedra of different sizes. There is also an intriguing tensegrity robot (Fig.9d) that consists of two linked tetrahedral frames and a few cables under tension between pairs of vertices in different frames [5].



(a) (b) (c) (d)

Figure 9: *Interlinked tetrahedra: (a) 'Star Tetrahedron' by Fitz-Gerald [4], (b) 3D-print of five symmetrically linked tetrahedra [11], (c) '15 Tetrahedra' by Kwan [7] (d) Tensegrity Robot [5].*

With the goal of trying to make a contribution to this domain of the art world, we selected a tetrahedral as well as a cube-based configuration that both have the promise to make impressive constructivist sculptures when realized at a large enough scale. In both cases, we looked for the geometric realization of a specific configuration that would constitute a local optimum in terms of the maximum thickness that could be used for the tubular edges. These configurations would then be rigid without the need for any additional elements to hold the two frames in their relative positions against one another. For the tetrahedral pair we chose configuration "4-4C1" with the maximal tube thickness of 7.16% of the edge length. For the cube we chose the D_3-symmetrical 6-6 linking shown in Figure 4 with a tube thickness of 6.68% of the edge length. We then also chose dramatic, yet balanced angles at which to mount the configurations. Our proposed sculptures are shown in Figures 10(a) and 10(b).


(a)

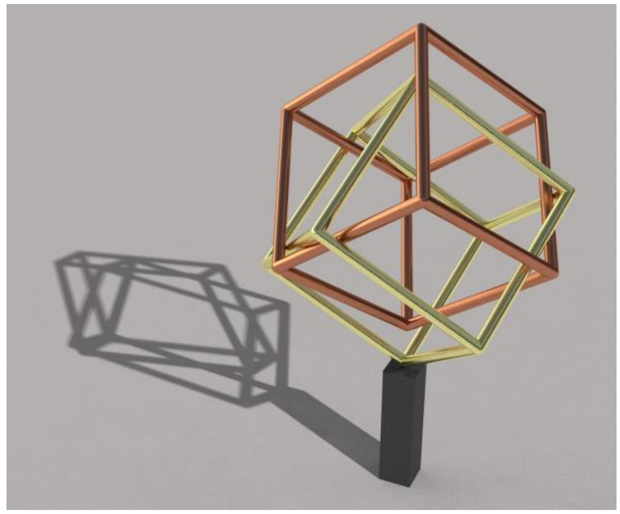

(b)

Figure 10: *New sculptures based on interlinked polyhedral frames: (a) two tetrahedra, (b) two cubes.*

8. Discussion and Conclusion

In a strict mathematical sense, the work of finding all linked pairs of two identical polyhedral frames is not finished until we can prove more formally that the link matrix is indeed an appropriate discriminator to tell us whether two configurations are different, and until we have a reliable way to prove that we have found all topologically different linkings for a particular polyhedron pair. While we believe that we have found all configurations for two tetrahedral frames, based on our rather extensive search and on a study of the evolutionary graph, we are less sure about our collection of linked cubes. A few more cube-frame linkages may be hiding in 6D-space. Moreover, there are still the linkings of the three other Platonic solids to be explored. We invite bright mathematical minds to help us in this quest.

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