Two Complementary Ways of Linking Math and Art

Carlo H. Séquin

CS Division, University of California, Berkeley; sequin@cs.berkeley.edu

I am a very visually oriented person. Geometry has been in my blood since high-school, when I was given a copy of Hermann Weyl's book "Symmetry" [12]. Formal mathematical proofs do not appeal to me until I can get some visualization model that supports that proof on an intuitive level. This is why I have been captivated by topics such as regular maps, graph embeddings, and mathematical knots. Corresponding visualization models have led to geometrical sculptures that convey an aesthetic message even to people who do not know the underlying mathematics. Conversely, abstract geometrical artwork by artists such as Brent Collins and Charles O. Perry have prompted me to discover the underlying mathematical principles and capture them in computer programs, which then produce more sculptures of the same kind.

For me, the interaction between art and mathematics flows in both directions. In one scenario, I may start out by trying to make a good visualization model for a mathematical concept. For example, I may want to show graphically how many different types of Klein bottles exist [8] that cannot be smoothly deformed into one another with a *regular homotopy* (this is a process that allows surface segments to pass through each other; but one is not allowed to make any cuts or introduce sharp creases). The models of some of the more intricate Klein bottles, which may be shown in a translucent manner by making their surfaces a partially transparent, gridded structure, may resemble constructivist geometrical sculptures [7]. As another example, I may want to show how the complete graph with twelve nodes (K12) can be symmetrically embedded in a handle-body of genus 6 (e.g., a doughnut with 6 holes); this may end up as an intricately painted surface with the looks of a piece of strange jewelry [6].

In the opposite scenario, I become fascinated by a special piece of abstract geometrical sculpture, and I will try to understand its basic topology and all its possible symmetries. Then I try to capture those insights in a parameterized computer program that can reproduce the basic geometry as well as make related versions of that shape. In the case of C. O. Perry's 18-inch sand-cast sculpture called "Star Cinder" [5], this surface is defined by a tangle of ten equatorial triangular loops that interlock with icosahedral symmetry, as described by Figure 8.16 in Alan Holden's book "Orderly Tangles" [4]. These ten loops then form the border curves of a soap-film-like surface. Seeing a picture of this sand cast, prompted me to model it and to make a 3D-print of this shape (Figure 1a). I also made models of "soap-film" surfaces based on other sets of interlinked equatorial loops, such as the Borromean rings [9] (Figure 1b), or surfaces suspended by a tangle of four triangular loops with cuboctahedral symmetry (Figure 1c).

(a) (b) (c)

Figure 1: *3D-prints of simple star-cinders: (a) icosahedral surface matching Perry's "Star Cinder;" (b)"soap-film" surface on the Borromean rings; (c) cubist arrangement with 4 triangular loops.*

This process then leads to an additional, more challenging quest. The simple "star cinders" mentioned above only fill a small outer part of the spherical volume spanned by these surfaces. How can I construct a similar type of surface that will fill the sphere volume more completely? Clearly, I need to start with a set of border curves that dive more deeply into the sphere and form paths that pass close to the sphere center. In a first step, I replace the triangular loops in Perry's "Star Cinder" with trefoil knots or with more complicated curves that encircle the center point more tightly [10]. In the example shown in Figure 2, each triangle loop has been replaced with three interlinked circular borders (which may be seen as (3,3)-torus knots). There are a total of 30 such circles, one each associated with every edge of the icosahedron. The challenge then was to construct a single 2-manifold "soap-film" surface between these 30 border curves that retains the full symmetry of the oriented icosahedron and has no self-intersections. This particular solution has three concentric, radially coupled shells, each with 20 triangular and 12 pentagonal openings.

Currently I am developing similarly structured "soap-film" surfaces with toroidal symmetry or based on spirally wound surfaces like a nautilus shell [1]. They offer an additional challenge over the spherical structures discussed above. The smaller, inner shells can no longer be obtained by a simple scaling operation. Each level now requires some individual geometrical design. Another technical challenge is to bring these "soap-film" surfaces closer to true, locally minimal surfaces. For the current designs I construct polyhedral approximations of the expected saddle surfaces and then use 3 to 5 levels of Catmull-Clark subdivision [3] as a smoothing operation. Software that can help in this challenge has been developed by Ken Brakke in his "Surface Evolver" [2].

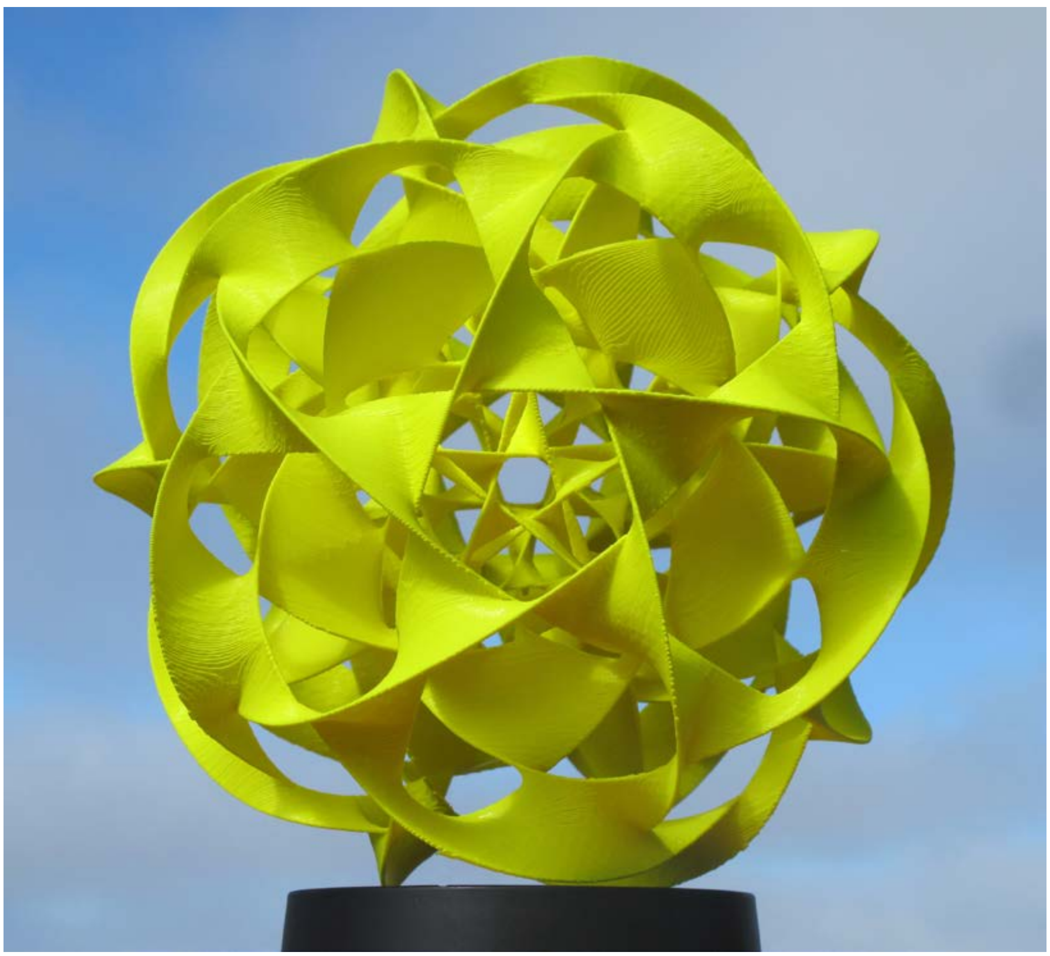

Figure 2: *3D-print of a 3-level star-cinder on an icosahedral tangle of 30 circular borders.*

While I have described two conceptually clean pathways going in opposite directions to create mathematical sculptures, the approach is often more "entangled". Take the example of a nautilus shell realized with multiple levels of nested and interlinked shells. The overall shape as well as the detail of the soap-film surfaces are clearly inspired by nature. The geometrical solution of linking shells at different levels in an attractive way emerges from the sculptural maquettes discussed above. Yet the basic idea of connecting border curves with "soap-film" surfaces has artistic roots (C.O. Perry) as well as a mathematical background in the form of Seifert surfaces [11].

Even though my sculptural designs have a high degree of symmetry and follow formal generating processes, they do not provide an answer to the looming, perennial question: When is something "Art"? This is still a highly subjective assessment – and probably always will be. How much complexity is intriguing, and when is it simply too much? When is symmetry an asset that increases the aesthetic value of a shape, and when is a judicious break in symmetry what turns a geometrical shape into a fascinating piece of art? In my work, I typically am not able to say explicitly how I judge the artistic value of a particular shape. I simply trust my intuition. But my judgement may change over time, and a decade after I created an object, I might no longer think that I had selected an optimal set of parameters.

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