Gosper Sculptures Revisited

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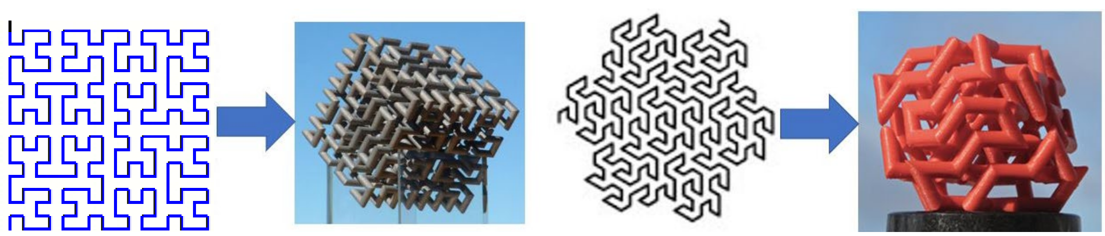
Abstract

The paper starts with a review of three hierarchical designs for tubular 3D sculptures that capture the style of the 2D Gosper curve: *Gosper-Balls*, *Gosper-Pole*, and *Gosper-Lattice*, and it discusses some of their shortcomings. It then analyzes three non-hierarchical approaches: *Gosper-Stars* with some internal branching, *Gosper-Onions* that use a small number of layered shells, and *Gosper-Shells* that map a Gosper-like path onto the surface of a regular polyhedron. The focus is on creating a modular polyline that connects nearest neighbor vertices on a regular lattice, while favoring the characteristic turning angles of 60 and 120 degrees of the 2D Gosper curve.

Introduction

Twenty years ago, I presented a recursive procedure to fill 3D Euclidean space with a single polyline inspired by the 2D Hilbert curve. A finite portion of this curve then made an attractive constructivist sculpture [3]. Now I am trying to do a corresponding construction starting with the 2D Gosper curve.

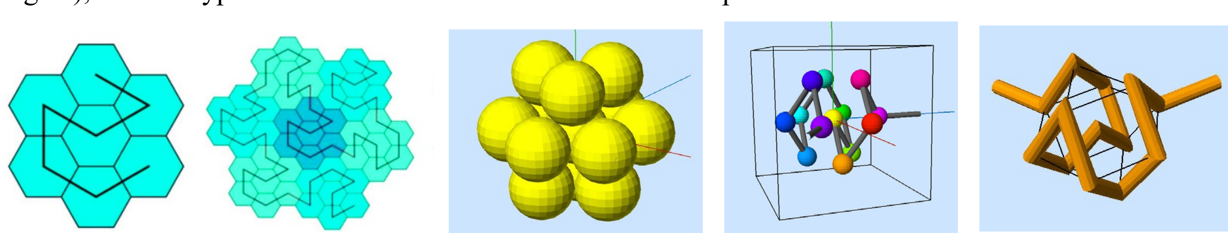


(a) (b)

Figure 1: *3D sculptures based on 2D space-filling curves: (a) Hilbert curve, (b) Gosper curve.*

A Basic 3D Gosper Configuration

In my 2021 FASE paper [4], I studied several ways to take the 2D Gosper curves [6] [10] [11] into 3D with the goal of making attractive sculptures. Rather than using the original Gosper curve [10], which is based on a recursive line-substitution, I used a variant of the Node Gosper curve [11], which is based on a recursive reuse of hexagonal tiles (Fig.2a,b). This recursive use of tiles in 2D has a natural analogy in 3D in a clump of 13 atoms in densest sphere packing (Fig.2c). This approach places the 13 atoms in this clump at the vertices and the center of a cuboctahedron, which is a subset of the FCC (face-centered cubic) lattice. The 13 atomic sites are connected with twelve segments between nearest neighbors, plus two more stubs that will connect to adjacent clumps (Fig.2d). This path, fattened into a tubular strand (Fig.2e), forms a type of modular element from which 3D sculptures can be constructed.



(a) (b) (c) (d) (e)

Figure 2: *(a, b) Two generations of the Node-Gosper curve. (c) Densely packed 13-atom clump; (d) a Hamiltonian path in this clump; (e) corresponding tubular strand.*

What counts as a valid 3D Gosper curve? There are different possible constraints, and they may be enforced with different priorities. For my explorations I have used the following criteria:

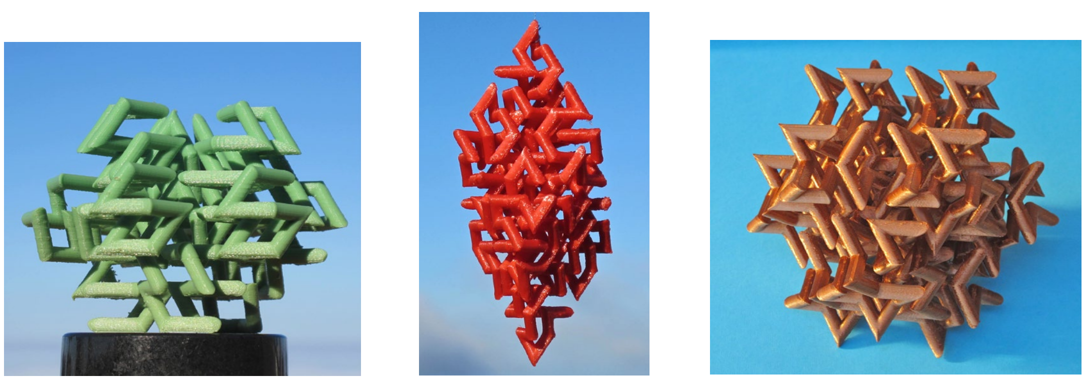
The result should be a single polyline with all equal-sized steps. All the vertices of this polyline should fall on the sites of a common 3D lattice, and they should occupy *all* the sites in some compact "island." Ideally, this island would have a roughly spherical shape, possibly with a fractal boundary. Locally, the polyline should exhibit the features of a 2D Gosper curve: claw-like shapes where the polyline primarily makes bends of 60° or 120°. Occasionally, bends of 90° or 0° are also acceptable. I would like the construction to be as isotropic as possible. This means that all the directions taken by nearest-neighbor connections should occur equally often. A more demanding goal might be to also find all turning-angle directions with the same frequency. (For the FCC lattice, there are only six possible directions for the polyline segments, but there are 24 possible directions for the 120°-turns.)

Ideally the polyline follows a recursive hierarchical structure. The sculpture then consists of some type of *clump* comprising *A* atoms, and groups of *A* such clumps are then joined into *super-clumps*. Recursively, *A* super-clumps form the next higher hierarchical element in the sculpture. The geometry of the path by which atoms are connected in the lowest-level clump are reflected in the way that clumps are connected to form a super-clump. Ideally, if the base-curve has just bending-angles of 60° or 120°, it should be sufficient to design two types of clumps that have their two connection points 60° and 120° apart, respectively. These two clumps could then be used with simple rigid-body transformations (including mirroring) to accommodate all possible nearest-neighbor connections between adjacent clumps. The same kind of transformations would also take care of all connections between super-clumps.

Unfortunately, a completely recursive definition of a 3D Gosper curve has not yet been found, – and probably does not exist [8]. On the other hand, if the main goal is to make attractive 3D tubular sculptures, this deficiency may not be too tragic – as will be shown in the next two sections.

Hierarchical Approaches for Making 3D Gosper Sculptures

My paper in the FASE track at SMI_2021 [4] describes various approaches to make finite 3D Gosper-like sculptures (Fig.3). Here is a brief review of three promising hierarchical constructions and a discussion of their shortcomings.

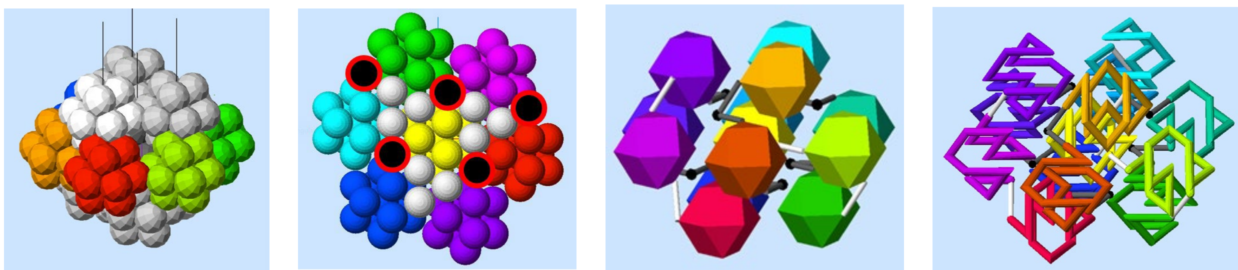


(a) (b) (c)

Figure 3: *Sculpture styles: (a) Gosper-Ball_175; (b) Gosper-Pole_216; (c) Gosper-Lattice_351.*

Gosper Balls

Gosper-Ball_175 (Fig.3a) is constructed from 13-atom clumps. Thirteen such roughly spherical clumps with serrated surfaces are packed densely (Fig.4a), so that in the equatorial plane of this assembly the seven equatorial atoms of seven clumps form the same interlocking pattern as shown in Figure 2b. However, this densest packing does not fill *all* the sites of the FCC lattice; one out of every 14 sites remain unfilled (Fig.4b). To remedy this flaw, six voids internal to this Gosper Island are filled with extra *filler-atoms*; these atoms are then visited in six of the thirteen transitions from one clump to the next (Fig.4c). This results in an overall strand that has little modularity and no symmetry at all. The connecting polylines in most of the 13 clumps are slightly different and had to be hand-crafted (Fig.4d).

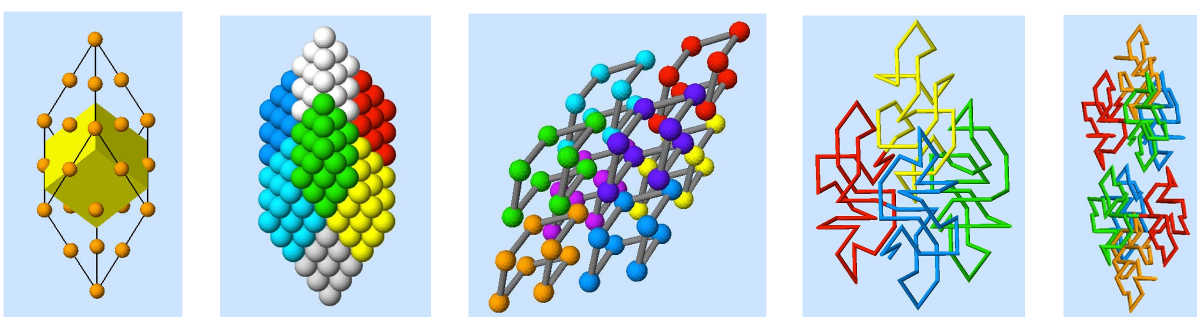


(a) (b) (c) (d)

Figure 4: *(a) Three layers of 13-atom clumps (side view); (b) void FCC lattice sites (top view). (c) Using filler-atoms (black) in the voids as stepping-stones; (d) resulting Gosper-like path.*

Gosper Pole

The FCC lattice can be obtained from a cubic lattice by stretching the latter by a factor of two along one of its space-diagonals (Fig.5a). This transforms cubic cells into rhomboid cells, and these cells tile 3D space seamlessly with no voids at any FCC sites (Fig.5b). If the rhomboid cell is filled with just eight atoms placed at its corners, the connecting Hamiltonian polyline is reminiscent of a deformed Hilbert Cube (Fig.5c), – more so than representing a Gosper curve. Therefore, in *Gosper-Pole_216* (Fig.3b), the rhomboid cells have been populated with 27 atoms, and eight such cells have been assembled into an overall rhomboid shape. Only four different connection strands had to be designed for the 27-atom clump (Fig.5d), and two such structures fit together with C_2 symmetry (Fig.5e). However, the resulting sculpture is quite anisotropic (Fig.3b), deviating notably from the round islands of the 2D Gosper curves.

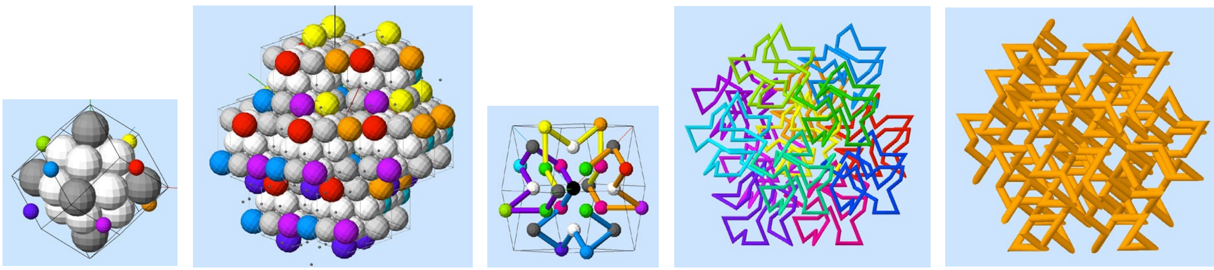


(a) (b) (c) (d) (e)

Figure 5: *(a) Diagonally stretched cube; (b) rhomboid tiling of 3D space; (c) eight 8-atom cells. (d) Four different paths in a 27-atom cell; (e) two such assemblies form Gosper-Pole_216.*

Gosper Lattice

A better and more isotropic cell that tessellates 3D space nicely is the rhombic dodecahedron (RD). Each such tile contains 19 internal FCC sites, where spherical atoms can be placed so that they mutually touch their 12 neighbors, plus an additional 24 atomic sites on its border – one each on every edge of the RD (Fig.6a). To prevent neighboring tiles from placing more than *one* atom into a particular site, only 8 of the 24 border sites must be associated with each RD tile. I was able to find an arrangement that places these border sites in such a way that the result is a 27-atom clump, where 13 of them can be arranged in a densest sphere-packing configuration without any overlaps and without any interstitial voids (Fig.6b). The strand in *Gosper-Lattice_351* (Fig.3c) then visits all 13*27=351 sites in a polyline path with equal step-lengths, while making mostly 60° and 120°-turns at every site (Fig.6d,e). However, in order to obtain one single-loop path, I had to design nine different clumps with different versions of a 27-segment Gosper path (Fig.6c) to make appropriate connections with the paths in two neighboring RD tiles, since the filler atoms makes these clumps anisotropic, and they can no longer be rotated freely into all the required orientations to make the proper connections to two neighbor clumps. This involved a rather tedious manual design approach with no appreciable clump-level modularity.



(a) (b) (c) (d) (e)

Figure 6: *(a) RD clump with 19 atoms; the 8 colored filler-atoms are on the edges; (b) thirteen 27-atom RD clumps in tightest sphere-packing. (c) One possible Hamiltonian path through a single clump; (d) a properly connected path through all 351 sites; (e) resulting tubular sculpture.*

Properties and Limitations of Finite-size 3D Sculptures

The shortcomings of the sculpture designs reviewed above prompted me to look for alternative approaches. To guide my efforts towards better designs, I re-examined some relevant criteria for making attractive 3D sculptures that capture the characteristics of the Gosper curve.

The ideal 3D Gosper sculpture consists of just a single contiguous strand. That strand is subject to gravitational forces. For my *Hilbert-Cube_512* sculpture (Fig.1a), if the strand were stretched out horizontally as much as possible, it would extend about 600 tube diameters and bend significantly. Even curled up inside the hull of a cube, it still experiences some sagging, and the whole sculpture feels kind of squishy. In sculptures with a longer loop the strand would sag so much that some tube segments placed above one another in the sculpture may touch. This limits the size of physical tubular sculptures to just a few hundred segments (depending on size and material), and it restricts us to just two hierarchical levels. This prevents a truly recursive approach; but, on the other hand, this restriction offers some advantages and additional design freedom for the construction of sculptures of finite size.

In a finite physical sculpture, I want the connectivity at the coarsest level of the clumps be a closed path, so that there are no free, open ends sticking out from the sculpture. On the other hand, the polylines running through the clumps at the lower hierarchical level must be open-ended, so that these clumps can connect to their neighbors. (This is the approach I used in *Hilbert-Cube_512*.) In a first new design approach, used in *Gosper-Star*, I go even further and no longer adhere to a strict 1-manifold curve, and I introduce some internal branching to provide mechanical support to counteract the force of gravity.

There is another reason why increasing the number of tube segments beyond some upper limit is counterproductive. All 3D Gosper curves I have been able to construct with reasonable spatial isotropy have a somewhat "chaotic" look. This is quite different from the structure of *Hilbert-Cube_512* (Fig.1a), where whole families of parallel tube segments form an orderly structure that can readily be perceived even in cubes with many hundreds of tube segments, and which would still be perceived as regular "textures" for cubes with many thousands of segments. In contrast, the Gosper-nature of a 3D tubular sculpture has to be conveyed in the outer few layers of a compact tubular cluster. The inner structure of the 3D sculpture almost gets in the way by distracting from a possible clearer view of any Gosper features on the outermost skin. Trying to deal with this issue has led to two types of sculptures designated as *Gosper-Onions* and *Gosper-Shells*, discussed in the two subsequent subsections. They replace the hierarchical structure of the sculpture with a more layered approach, where the characteristic features of the 2D Gosper curve are realized in a bent form on the outer curved surfaces.

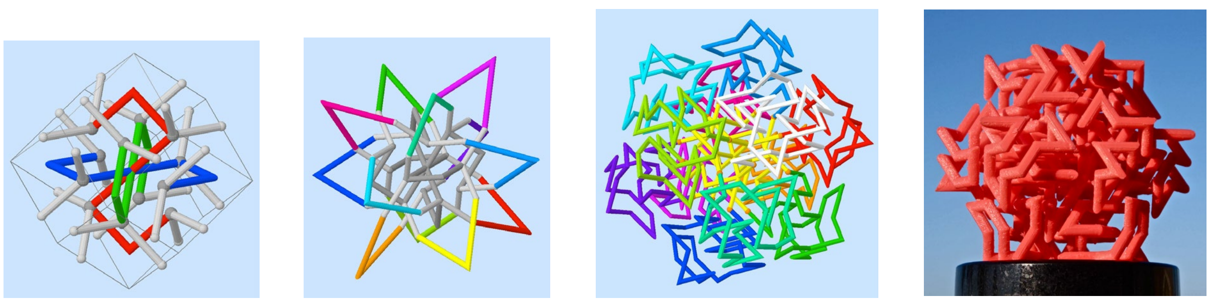
Gosper Sculptures that Ignore some of the "Rules"

By focusing on some of the important visual aspects of finite 3D-sculptures and ignoring some of the "rules" that originated from an initial goal to find a recursive, space-filling 3D Gosper curve [8], we gain additional design freedom for designing finite Gosper sculptures.

Gosper-Stars

When packing thirteen compact clumps into a densest sphere-packing configuration, the central clump plays a minor role with respect to the visual qualities of the overall sculpture, since it is much less visible than the twelve clumps surrounding it. Thus, I allow it to be structurally different from the other twelve clumps. Here I even give up the notion that its strand should be a simple 1-manifold, and I introduce some internal branching to gain additional mechanical stability for the whole sculpture.

In *Gosper-Star_354* (Fig.7d), the outer clumps are based on the RD-cell used in *Gosper-Lattice*. But the central clump has a tree structure with its root in the center, from where it is branching out into a tree with 24 leaves (Fig.7a). Every one of the 12 outer clumps forms a single 1-manifold loop that connects to two of these leaves (Fig.7b). The central tree exhibits 6-fold D_3 symmetry; but the outer loops cannot all be exactly the same, because there are some shared FCC sites on the boundaries between neighboring clumps. I found that with just three slightly different versions of these loops (Fig.7c), I can cover all FCC sites in the island covered by the final sculpture.



(a) (b) (c) (d)

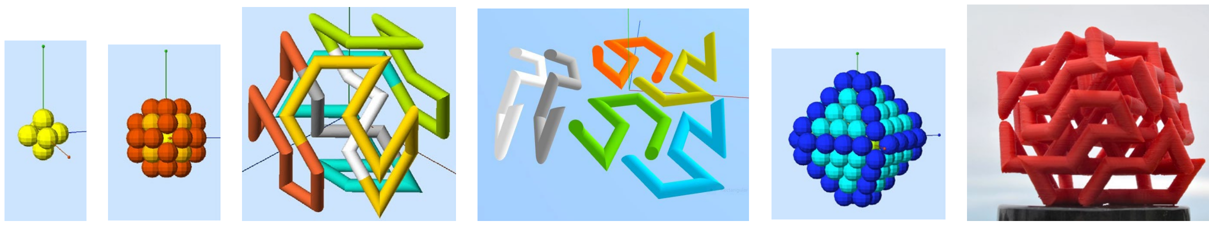
Figure 7: *(a) Central cluster with 24-way branching; (b) conceptual view of 12 attached outer loops; (c) fully instantiated outer paths; (d) resulting tubular sculpture: Gosper-Star_354.*

We can push the non-hierarchical approach even further. There is no need to have a central clump that is equal in size to the peripheral Gosper clumps. We can focus on designing such an outer clump to some practical size that we are willing to accommodate in an actual sculpture. Then we might tightly pack together only *four* such clumps in a tetrahedral configuration, or perhaps *six* such clumps in an octahedral assembly. Then we use all the central FCC lattice sites in the space between the peripheral clumps to form a tree structure that connects to all the ends of the outer Gosper paths. There are many more design avenues to be explored.

Gosper-Onions

In this approach, I give up the hierarchical structure of the sculpture based on multiple clumps. Instead, I focus on creating a single dense clump that on its outer, most visible surface displays some of the characteristic claw-like elements of the 2D Gosper curve (Fig.2a,b). With this goal in mind, I build this clump as an assembly of spherical layers like the skins in an onion.

In *Gosper-Onion_38*, I start with a central octahedron with six atoms (yellow) (Fig.8a). In a second shell I place 24 atoms (red) on the vertices of a truncated octahedron and 8 more atoms (orange) in the centers of the eight hexagonal faces (Fig.8b). This second shell allows me to place four Gosper claws onto four of the hexagonal faces and use the remaining sites in this shell to make connections to the internal octahedral sites. The third shell of a *Gosper-Onion* would comprise 102 atomic sites (Fig.8e).



(a) (b) (c) (d) (e) (f)

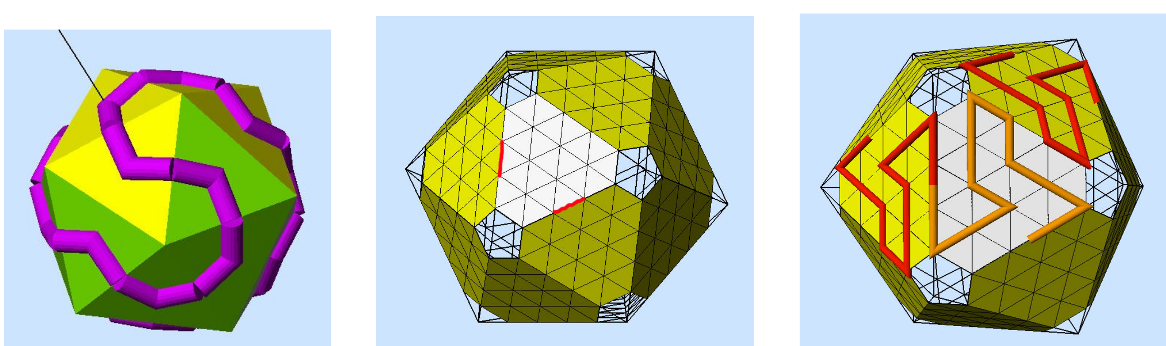
Figure 8: *(a) Octahedral core, 6 atoms; (b) 2nd shell, 32 atoms. (c) Path of the 2-level Gosper-Onion; (d) assembly based on six parts. (e) 3rd shell, 102 atoms; (f) a 3-level Gosper-Onion sculpture.*

To gain stability and some symmetry in a 2-level *Gosper-Onion*, I break the outer shell into two identical halves and connect their ends individually to two branches, each using three sites of the octahedron (white, grey) (Fig.8c). The result is a small sculpture that clearly conveys a Gosper motif on its surface.

Because of its low complexity, it is a good candidate for a large-scale construction from 38 mitered tubular elements. Chris Ohler is contemplating to realize this sculpture with metal tubing. To shape this assembly as close to the designed symmetry as possible, it is advantageous to build the sculpture in six parts. Four parts comprise the outer, planar claw elements connecting the seven FCC sites in one of the hexagonal faces, plus two more tubular segments sticking out in the third dimension (red) (Fig.8d). The other two identical parts (yellow) realize the internal connections through the octahedral sites. These latter two parts can be positioned initially in a temporary jig, so that the four outer parts can be attached to them. Figure 8d shows the six parts laid out for 3D printing of a small model to test the assembly procedure. Figures 8e shows the 3rd onion shell layer with 102 atoms, and Figure 8f shows a small model of a 3-level *Gosper-Onion* sculpture.

Gosper-Shells

Since the outer layer of *Gosper-Onion_38*, displays the Gosper claws so convincingly, I decided to explore what would happen, if the whole sculpture consists of just this outermost layer. One way to do this is to form a roughly spherical shell in the form of a truncated icosahedron. The 20 hexagonal faces would then be filled with hexagonal cut-outs from a 2D Gosper curve. To obtain a closed 1-manifold strand, I construct a Hamiltonian circuit on a dodecahedron, which informs me how the 20 hexagon patterns must connect to one another (Fig.9a) to form a single loop. Ideally all 20 faces should carry the same pattern.

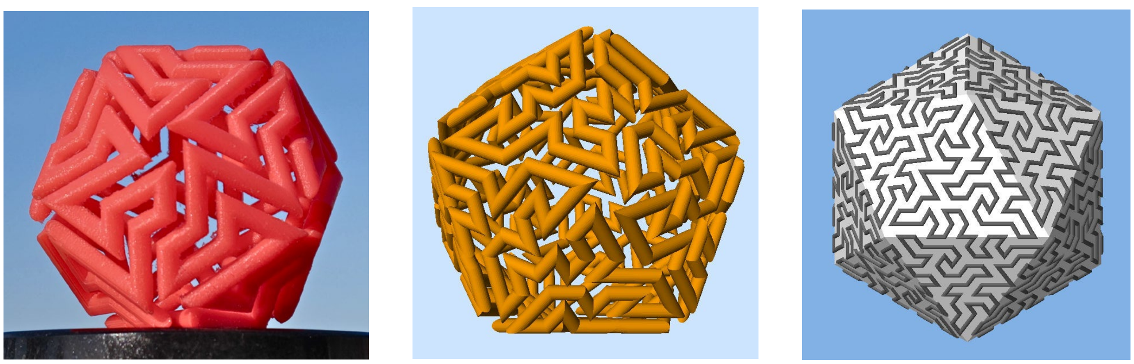


(a) (b) (c)

Figure 9: *(a) a Hamiltonian path; (b) subdivided icosahedron faces; (c) a possible Gosper pattern.*

Finding an appropriate pattern turned out to be more difficult than expected. The 2D Gosper curve cannot just be trimmed to fill the icosahedral face nicely while providing the needed connectivity across the icosahedron edges to the neighboring faces. With the goal to construct a useful polyline that covers the surface with uniform density, I subdivide each triangular icosahedron face by a factor 5 or more, and I truncate the icosahedral corners just to the nearest subdivision vertices (Fig.9b). The vertices lying on the icosahedron edges must be assigned to only *one* of the two adjacent faces, except for the few vertices where the Gosper strand transitions from one face to one of its neighbors. Optimally, these transitions occur in the middle of an edge, where they involve two vertices and a shared poly-line segment running along the edge (Fig.9b). When observing all these constraints, it is a challenging task to devise a poly-line path that visits all the subdivision vertices and displays convincingly a Gosper-like characteristic.

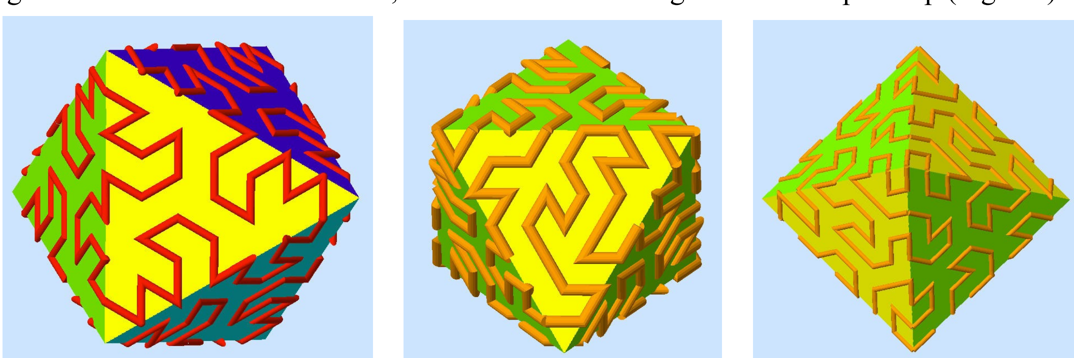
Subdivision-level 5 yields the minimal number of sites that allowed me to construct an acceptable polyline (Fig.9c); but it has some tubular stretches of length 3 along the icosahedron edges. The resulting sculpture is shown in Figure 10a. Subdivision-level 7 yields a more desirable result (Fig.10b) without any 3-unit straight segments. Figure 10c shows a pattern for subdivision-level 11. This curve has 1200 segments, and this may be too much for an actual physical sculpture, but it could make a nice, engraved design on the surface of a solid icosahedron. Any of these subdivided triangular tiles could also readily be used to wrap a Gosper curve around an octahedron, or even a tetrahedron; but I wanted the overall shape to be as round as possible.



(a) (b) (c)

Figure 10: *Gosper Icosahedra: (a) subdivision level 5; (b) subdivision level 7; (c) level 11.*

However there are some interesting possibilities when using an octahedral shape. Dan Bach [1] designed a nice, serrated edge-path that nicely captures the Gosper look. By cutting off the valence-4 corners in his octahedral edge-graph and reconnecting the 12 zig-zagging edges into an Eulerian circuit, I have obtained the single-loop Gosper curve shown in Figure 11a. Jeffrey Ventrella used the octahedral shape differently. He placed a roughly hexagonal patch from the central part of a 2D Gosper curve onto four octahedral faces in a *tetrahedral* manner and let three peripheral zones of each patch fold over onto the neighboring triangles [9]. By slightly modifying the Gosper patches, I could connect them selectively through four of the octahedral corners, so that I obtained a single closed Gosper loop (Fig.11b).



(a) (b) (c)

Figure 11: *(a) 12 serrated octahedron edges connected into a single loop. (b) Four "bent" Gosper patches on subdivision level 7; (c) four bent patches on subdivision level 8.*

Figure 11c shows the same approach on an octahedron with subdivision level 8. The two octahedron vertices that do *not* carry any patch-to-patch connections are shown at the left and right extremes.

Summary and Conclusions

My exploration of tubular sculptures that aim to reflect the nature of the 2D Gosper curves emerged from an original quest to find a recursive, space-filling 3D Gosper curve that fills all of 3D space in a regular manner [8]. Thus, it was natural that my first experiments focused on hierarchical assemblies of identical clumps. In this paper, I focus on three new approaches to construct finite-size, compact sculptures that are rather distinct from that original goal. As in *Hilbert-Cube_512*, I still prefer *closed* curves for my sculptures, with no open ends sticking out awkwardly. Thus, the connectivity at the coarsest level is already inherently different from the connectivity within each clump. In *Gosper-Star_354,* I go one step further and allow some internal (essentially invisible) branching to obtain more overall stability.

In the *Gosper-Onion* and *Gosper-Shell* structures, I forego any hierarchical clump structure and construct just a single cluster with an outermost layer optimally designed to exhibit the character of a 2D Gosper curve. What I retain in all new designs is the fact that the polyline makes unit-length steps on a regular lattice, with a high preference for bending angles of 60° and 120°, and that the curve moves through 3D space in a fairly isotropic manner. I believe that some of the results allow for making attractive small models as well as large constructivist tubular sculptures.

Acknowledgements

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References

[1] D. Bach, “Zig-Zag Octahedron.” – gallery.bridgesmathart.org/exhibitions/2021-joint-mathematics-meetings/dansmath

[2] T Chen, “JIPCAD.” Tech-Report (EECS-2021-125) May 14, 2021. – https://www2.eecs.berkeley.edu/Pubs/TechRpts/2021/EECS-2021-125.html

[3] C. H. Séquin, "Hilbert Cube 512." Artist's Sketch, SIGGRAPH'06, Boston, July 30 - Aug. 3, 2006.

[4] C. H. Séquin, "3D Gosper Sculptures." SPM - SMI_FASE, Hyperseeing, Oct. 2021 – http://people.tamu.edu/~ergun/hyperseeing/2021/FASE/sequin2021.pdf

[5] J. Smith. “SLIDE design environment.” (2003). – https://people.eecs.berkeley.edu/~ug/slide/

[6] V. Uher, P. Gajdo, V. Snasel, “Towards the Gosper Space Filling Curve Implementation.” 2017 3rd IEEE International Conference on Cybernetics. – https://www.researchgate.net/publication/318575433_Towards_the_Gosper_Space_Filling_Curve_Implementation

[7] Ultimaker. “Reliable 3D printers.” – https://ultimaker.com/3d-printers

[8] J. Ventrella. “Is there a 3D Gosper curve?” – https://spacefillingcurves.wordpress.com/2016/09/27/is-there-a-3d-gosper-curve/

[9] J. Ventrella. “Gosper curve on an octahedron” Show and Tell at the BAAM meeting, Jan. 15, 2022.

[10] Wikipedia: “Gosper curve” – https://en.wikipedia.org/wiki/Gosper_curve

[11] Wolfram MatWorld: “Peano-Gosper Curve.” – https://mathworld.wolfram.com/Peano-GosperCurve.html