K_{12} and the Genus-6 Tiffany Lamp

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Abstract

The complete graph with 12 nodes, K_{12}, is mapped crossing-free onto a genus-6 surface with the highest degree of symmetry possible. The 66 edges of the graph partition this 2-manifold into 44 three-sided regions, which may then be colored differently. If this shape is illuminated from within, assuming translucent facets, the rendering of a genus-6 “Tiffany Lamp” can be obtained. A corresponding development can turn $K_{4,4,4}$, the dual of Dyck’s graph, with 12 nodes and 48 edges, into a highly symmetrical genus-3 “Tiffany Lamp”.

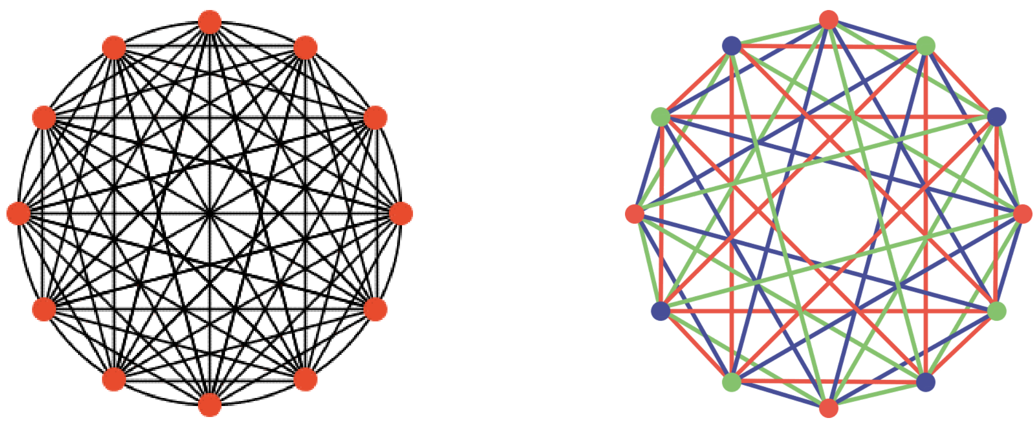

Figure 1. *Highly interconnected graphs: (a) K_{12} graph, (b) $K_{4,4,4}$ tripartite graph.*

Introduction

Highly-interconnected graphs cannot be drawn in the plane without crossings between some of their edges. However, if we choose a manifold of suitably high genus, crossing-free embeddings can be achieved. K_{12}, the complete (fully connected) graph with 12 nodes has 66 edges (Fig.1a). If we use these to define an oriented closed triangulated 2-manifold with 44 three-sided facets [2], then we can readily use Euler's formula to see that a surface of genus 6 is required to properly embed it:

$$\# \text{Faces} - \# \text{Edges} + \# \text{Vertices} = 2 - 2*\text{genus}$$
$$44 - 66 + 12 = -10 = 2 - 2*6$$

Actually, there are 59 topologically different ways in which this graph can be embedded in a genus-6 surface [1]. Thus it is not at all obvious what geometry should be chosen for that surface and for the embedding of the graph, in order to make the result most readily understandable and visually pleasing. Bokowski used an arrangement with D_3 rotational symmetry and built an intriguing physical model out of flexible gooseneck conduit sections [3]; but for most people it is difficult to see the triangular facets in this model (Fig.2). A similar problem is posed by the tripartite graph $K_{4,4,4}$, the dual of Dyck's graph [4], which is a subset of K_{12}. Each node is only connected to 8 neighbors (Fig.1b), and the corresponding triangulated 2-manifold has 48 edges and 32 three-sided facets. It requires a genus-3 surface to be properly embedded. We have tackled the challenge to make pleasing, highly symmetrical, easy-to-understand models for these two manifolds.

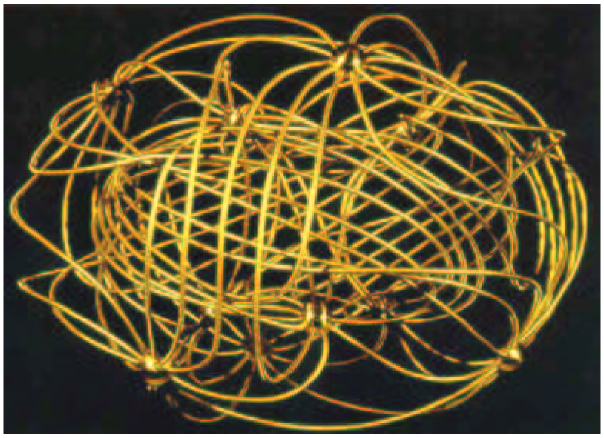

Figure 2. *Bokowski's gooseneck model[3] of the K_{12} graph.*

Exploiting Symmetry

A high degree of symmetry makes understanding of the model easier. The many sides of the model that are hidden from view can then be inferred by what is visible on the side facing the viewer. Thus a first step is to choose a surface with the proper symmetry group. For the complete graph K_{12} a surface of genus 6 is required, corresponding to a sphere with six handles. The appropriate symmetry is that of the tetrahedron; and the six handles can be arranged like the six edges in this 3D simplex (Fig.3).

Next we need to place the 12 nodes into symmetrical positions on this surface. There are several possible solutions: three nodes each around each corner of the tetrahedron, or around each opening corresponding to a tetrahedral face; or two vertices on each of the six edges. Preferably, the nodes should be placed into regions of high negative Gaussian curvature, to make available as much room as possible for the emerging eleven edges to spread out and take off towards all the other nodes that they need to connect to. Figure 3a shows the positions that we have finally selected, and Figure 3b shows a first physical mockup of this graph embedding. The symmetrical arrangement of crossing-free edges has been found by trial and error.

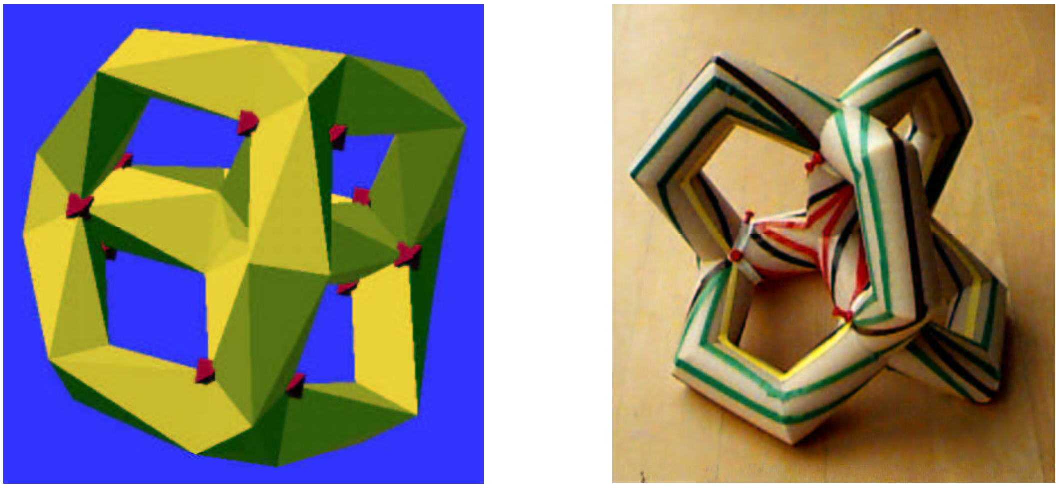

Figure 3. *(a) A polyhedral surface of genus 6 with the 12 node positions indicated.*
(b) A physical mock-up of the K_{12} model with tetrahedral symmetry.

For an embedding of the $K_{4,4,4}$ graph, a surface of genus 3 is sufficient. Such a surface also exists with tetrahedral symmetry; it is Klein's surface corresponding to a tetrahedral frame. Again there is a challenge to find good locations for the twelve nodes of the graph. But in this case there is an additional consideration. For every node there are three others to which there are no direct connections; thus we must consider carefully how to arrange these subsets of nodes.

After some searching we found what we believe to be the optimal placement. The subgroups of four nodes each that are not connected to each other are placed onto the D_2 symmetry axes of the surface (Fig.4a). Thus each arm of the surface, corresponding to an edge of the tetrahedral frame, carries two vertices, one on the inside and one on the outside. A physical model of the genus-3 Klein surface has been built on a rapid prototyping machine [6]. Nodes, edges, and facet colorings have then been painted by hand onto this model (Fig.4b).

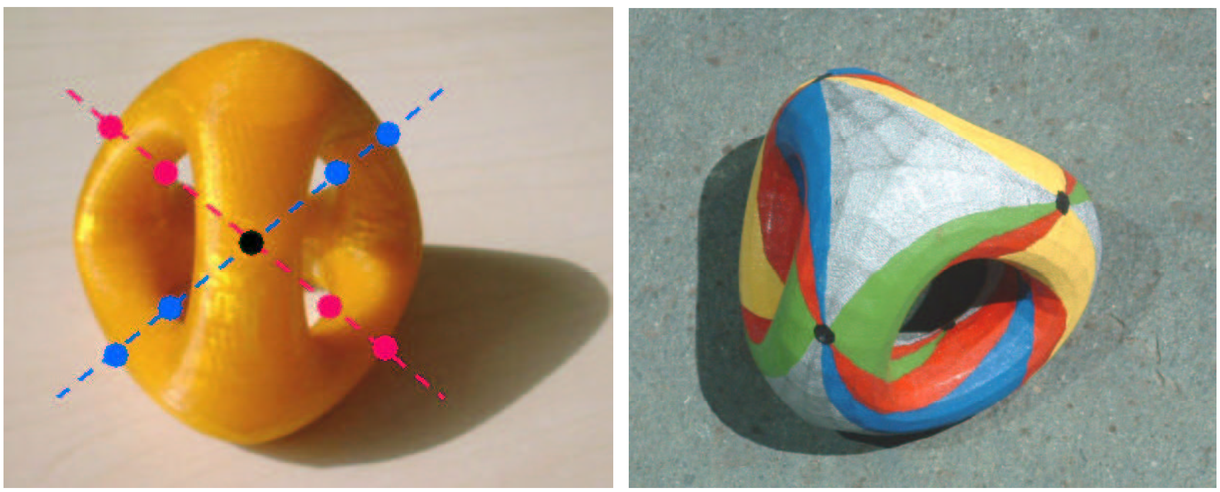

Figure 4. *(a) Klein's surface of genus 3 with tetrahedral symmetry, showing D_2 symmetry axes; (b) a rapid-prototyping model with the embedded $K_{4,4,4}$ graph painted on it.*

Smooth Curves on Smooth Surfaces

In order to obtain perfectly smooth, computer-generated objects of these two objects, we created parameterized subdivision models of the two surfaces discussed, and adjusted their parameters to obtain the most pleasing-looking shapes while maintaining the desired overall tetrahedral symmetry.

The edges between pairs of nodes were then modeled as "pseudo-geodesics", i.e., smooth curves with geodesic curvature that varies linearly with the arc-length measure. This gives us just enough degrees of freedom to be able to specify starting and ending directions for every curve; thus we can spread out reasonably uniformly all the edges emerging from a common node, without making the paths too convoluted. These curves were defined as crude poly-lines on a low-order, polyhedral approximation of the subdivision surface. Surface and curves were then refined together. After each additional subdivision step, the vertices defining each curve segment were optimized in their placement so as to approach the desired linear variation of their geodesic curvature.

The optimized, pseudo-geodesic edges were used to carve up the original subdivision surface into three-sided regions, which were then assigned different facet colors.

Final Renderings

These geometrical models then formed the basis for the computer renderings shown in Figure 5. The genus-3 manifold was modeled as a highly translucent surface made of colored glass, while the edges were modeled as thin, black cylindrical sweeps reminiscent of the lead fillings in a classical Tiffany lamp. Four ellipsoidal “light bulbs” were placed into the key junctions of this model, and the rendering was generated in day-long ray-tracing run with the program Radiance [5].

For the genus-6 surface the depth complexity of an image with fully translucent surfaces started to become overwhelming and confusing in a static 2D image. An opaque, but specularly reflecting, metal-like surface was thus chosen for the computer rendering of this virtual object (Fig.5b).

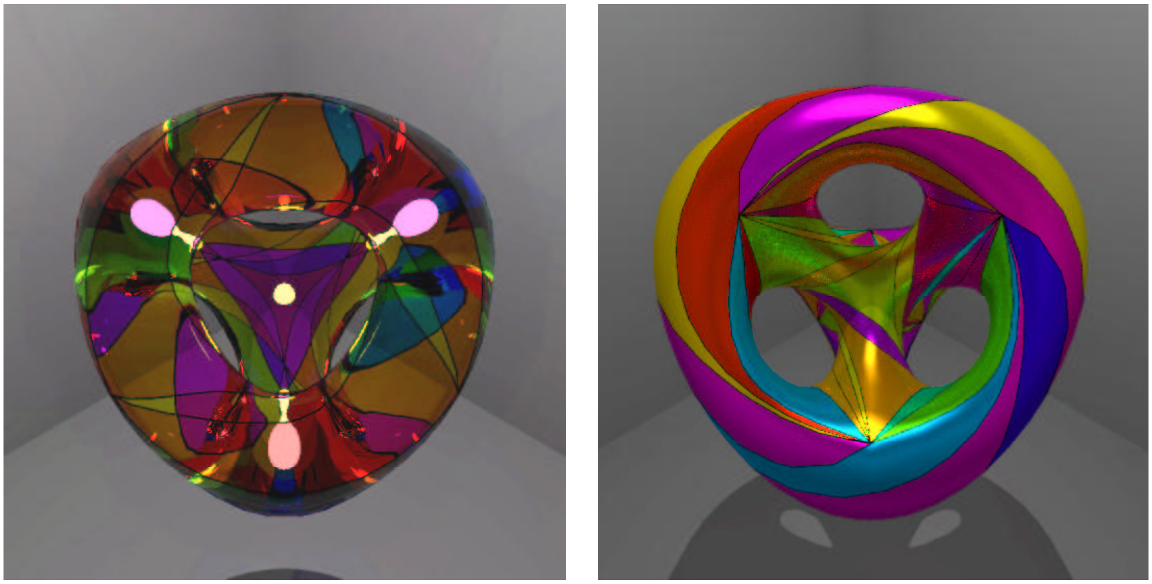

Figure 5. *Renderings of the virtual “Tiffany Lamps”: (a) Translucent surface of genus 3, (b) Metallic surface of genus 6.*

References

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