“Viae Globi” - Pathways on a Sphere

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Abstract

Sculptures by Naum Gabo or Robert Engman (Fig.1) contain as a key geometrical element a highly curvy, symmetrical, closed path on the surface of a sphere. For instance, in one of Engman’s pieces, this path is formed by the outer edge of a large ruled surface that is completely contained within the sphere, radiating inwards towards the center. Brent Collins has picked up this kind of a curve on a sphere and has swept a “C”-shaped cross section along it, resulting in a piece called *Pax Mundi*. We have developed a parameterized computer program that will form sculptures comprising such a meandering closed pathways on a sphere. Special CAD tools have been developed to construct the loopy generator curves along which the user can then sweep selected, parameterized cross sections to form a 3D solid object. Various parameters of the pathway and of the chosen cross section can be fine tuned with interactive sliders to maximize the aesthetic appearance of the resulting sculpture. Small scale models of a dozen different shapes from this sculpture family have been fabricated on a Fused Deposition Modeling machine.

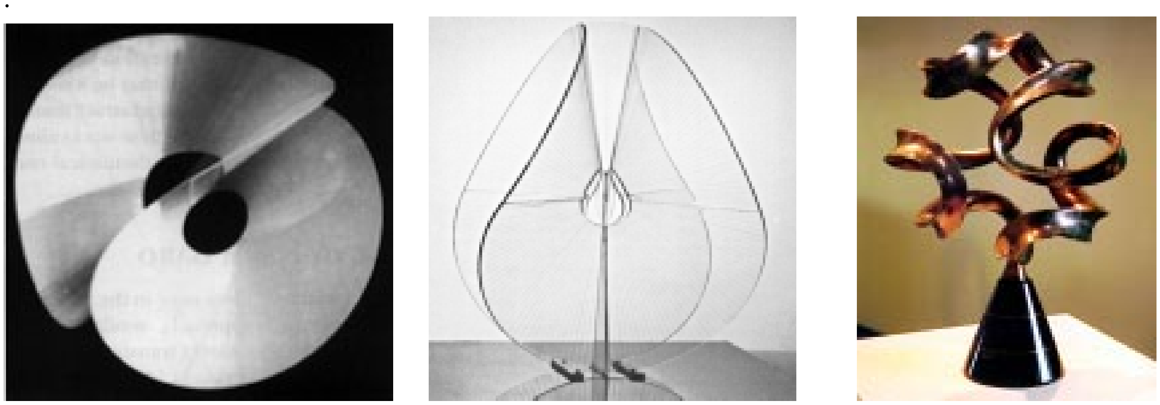

Figure 1: *Sculptures by Engman [11], Gabo [5], Collins [2] that served as inspiration for this paper.*

1. Computer-Aided Sculpture Design

Computers have become pervasive design aids. They not only serve engineers, chip designers, and architects, they also have become useful to product designers who create pleasing freeform shapes and, more recently, to artists who create abstract geometrical sculptures. My own interest in computer-assisted design of geometrical sculptures dates back to 1995. It emerged as part of a close collaboration with Brent Collins [1,8]. His work can be grouped into cycles that have a common recognizable constructive logic to them, and at the same time an aesthetic beauty that captured my attention immediately when I first saw photographs of his work in *The Visual Mind* [4].

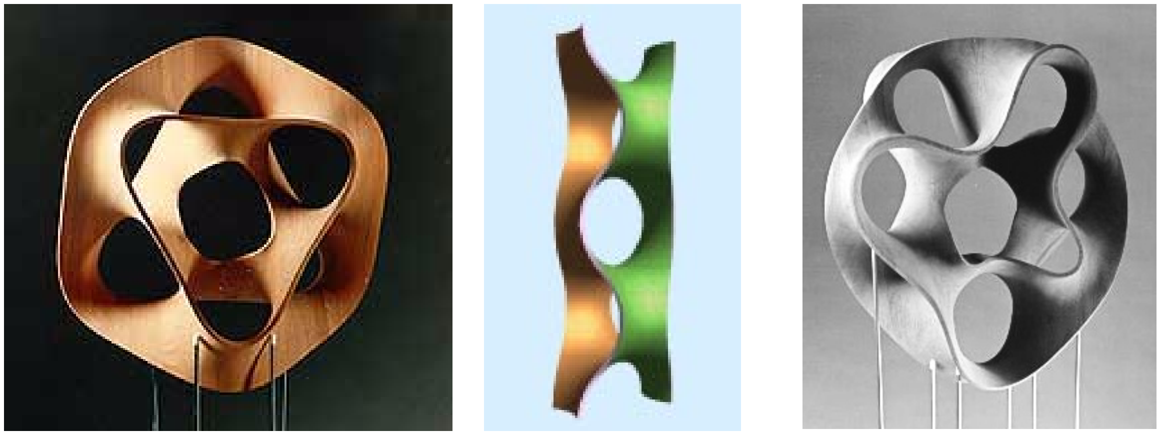

Figure 2: *(a) Collins' "Hyperbolic Hexagon," (b) Scherk tower, (c) Collins' "Hyperbolic Heptagon."*

My interaction with Brent Collins was triggered by images of his *Hyperbolic Hexagon* (Fig.2a), which can be understood as a toroidal warp of a six-story segment of the core of Scherk's second minimal surface [7] (Fig.2b). In our very first phone conversation, we raised the question of what might happen, if one were to take a seven-story segment of such a chain of cross-wise connected saddles and holes, and bend it into a circular loop. We realized that the chain would have to be given an overall longitudinal twist of 90° before its ends could be joined smoothly. We further envisioned that interesting things might happen in this process: the surface may become single-sided, and its edges could join into single continuous edge, forming a torus knot.

Since neither of us could visualize exactly what such a construction would look like, we both built little mock-up models from paper and tape (Séquin) or from pipe segments and wire meshing (Collins). In subsequent phone discussions, we expanded the scope of this paradigm. We asked ourselves, what would happen, if we gave the Scherk tower (Fig.2b) a stronger twist of, say, 270°, or of any additional 180° that allow the ends of the saddle chain to join smoothly. Or, what would a sculpture look like that used monkey saddles, or even higher-order saddles, rather than the ordinary (biped) saddles of the original *Hyperbolic Hexagon*?

Constructing a realistic maquette of these relatively complex structures, precise enough for aesthetic evaluation, can be a rather labor-intensive process. At that time, our ideas were coming forth at a rate much greater than what we could possibly realize in physical models. This lead me to suggest the use of the computer to make visualizations of these various shapes that would be good enough to judge their aesthetic qualities and to determine which ones might be worthwhile to realize as full-scale physical sculptures. I decided to develop a special-purpose computer program that could readily model these toroidal rings of Scherk's saddle chains, as well as all the generalizations that we had touched upon in our discussions. This lead to the *Sculpture Generator I* which allowed me to create all these shapes interactively in real time by just setting some parameter values on a set of sliders. [9].

In the meantime, Collins had built the *Hyperbolic Heptagon* (Fig.2c), the twisted seven-story ring that we had first discussed on the phone. This two-foot wood sculpture showed us the potential of this paradigm of toroidal loops of saddle chains, and encouraged us to make additional sculptures of potentially much higher complexity. However, such sculptures would require more help from the computer than just the power of previewing the completed shape. Thus I enhanced my program with the capability to print out full-scale templates for the construction of these sculptures. The computer slices the designed geometry at specified intervals, typically 7/8 of an inch, and produces construction drawings for the individual pre-cut boards from which the gross shape of the sculpture can then be assembled. Collins still has the freedom to fine-tune the detailed shape and to sand the surface to aesthetic perfection.

This eventually led to our first joint construction the *Hyperbolic Hexagon II* which features monkey saddles in place of the original biped saddles [10]. It is possible that Collins could have created this shape on his own without the help of a computer. However, our next joint piece, the *Heptoroid*, a much more complex, twisted toroid, featuring fourth-order saddles [10], would definitely not have been feasible without the help of computer-aided template generation.

2. Capturing a Paradigm

The collaboration with Brent Collins has added a fascinating new component to my professional life. Previously I had been developing CAD tools for more than a decade. In most cases, the task to be solved had been reasonably well defined: for instance, finding the optimal placement of the building blocks of an integrated circuit or extracting the connectivity in a given layout. Our group then had to find the best way of obtaining a solution to the given task. In my interaction with Collins, an important new component was added up front: I had to figure out what it was that I wanted my sculpture generator program to produce. This means that I first have to see a general underlying structure in a group of similar pieces in Collins' work and extract a common underlying paradigm that can be captured in precise enough terms to be formulated as a computer program. That by itself is an intriguing and creative task. Moreover, if the paradigm is captured in a general enough form, it can then be extended to find additional beautiful shapes that had not yet been expressed in Collins' sculptures.

The question arises, whether a commercial CAD tool such as *AutoCAD*, *SolidWorks*, or *ProEngineer* would have been adequate to model Collins' sculptures. Indeed, with enough care, spline surface patches and sweeps could have been assembled into a geometrical shape that would have matched one of Collins' shapes. But this approach would be lacking the built-in implicit understanding of the constructive logic behind these pieces, which then could be generalized and enhanced to produce many more sculptures of the same basic type. For that I needed stronger and more convenient procedural capabilities than these commercial CAD tools had to offer. I chose *C, C++*, and *OpenGL* as the programming and graphics environment, and originally *Mosaic* and later *Tcl/Tk* for the user interface, since my students had already developed many useful components needed in such a system, such as an interactive perspective viewer with stereo capabilities.

Capturing a sculpture as a program forces me to understand its generating paradigm. In turn, it offers precise geometry exploiting all inherent symmetries, as well as parametric adjustments of many aspects of the final geometry. The latter turns out to be the crux of a powerful sculpture generator. If I build too few adjustable parameters into my program, then its expressibility is too limited to create many interesting sculptures. If there are too many parameters, then it becomes tedious to adjust them all to make good-looking shapes. Figuring out successful dependencies between many different dimensions in these sculptures and binding them to only a few powerful adjustable sliders is the intriguing and creative challenge.

In practice it turned out that for almost every sculpture group that I have tackled so far, a new program had to be written. These programs are my virtual, constructivist sculpting tools. In the last couple of years, this virtual design environment has become more modular thanks to the *SLIDE* program library [13] created by Jordan Smith and enhanced with many useful modules for creating freeform surfaces by Jane Yen. Once a new program starts to generate an envisioned group of geometrical shapes, it often will take on a life of its own. In a playful interaction with various sliders that control the different shape parameters, and by occasional program extensions, new shapes are discovered that were not among the originally envisioned geometries. In this process the original paradigm may be extended or even redefined, and the computer thus becomes an active partner in the creative process of discovering and inventing new aesthetic shapes [10]. This paper describe this process for a new family of sculptures inspired by Brent Collins' *Pax Mundi*. I call this new series *Viae Globi*, since the shapes are reminiscent of the curvy pathways of a Swiss alpine road stretching around large portions of a sphere.

2.1 Rapid Prototyping

In the last few years, an exciting development has taken place: rapid prototyping of small-scale maquettes of free-form shapes has become widely available and (almost) affordable. Several different processes have reached commercial maturity [6] and are offered by many service bureaus. Through those services, one can obtain 3D visualization models or prototypes for an investment casting process for a couple of hundred dollars for a 3-4 inch piece; less expensive processes are under active development.

Such maquettes are the final visualization proof on which one can study the interaction of light and shadows and explore the shape through the sense of touch. They also provide an extremely useful reference during the construction of an actual full-size sculpture. Alternatively, his technology allows one to build a whole family of little sculptures at an affordable price for exhibition in a gallery or in an art-math road show [14]. Because the computer models of such sculptures are readily scalable, one can also use maquettes of appropriate sizes, possibly after some additional work-over by a skillful craftsman, to make bronze casts for sale in museum shops.

3. Periodic "Viae Globi"

This series of sculptures and its underlying procedural description was inspired by Brent Collins' *Pax Mundi* and by sculptures by Naum Gabo and Robert Engman which also exhibit such sweeping, meandering curves on the surface of a sphere. The latter curves are simpler, so I will discuss them first.

3.1 Gabo Curves

Gabo's or Engman's original realization of such curves (Fig.1) can be understood as two full waves wrapped around the equator of a globe, creating two pairs of lobes, two of which almost join at the North pole and two at the South pole, respectively (Fig.3a). My first attempt at modeling such a path was employing two periods of a sine wave drawn on a Mercator map of the world, where the wave amplitudes would be suitably modulated in height to produce the desired patterns at the poles. However, because of the large distortion at the poles of this global map, the shape of the curve needs additional controls, in particular to define the widths of the lobes (Fig.3b). To keep the curve smooth and well-rounded, I used a closed cyclic B-spline of 4th-order, and later even 5th-order, defined by three or four control points per half-wave. While three control points on as sphere would offer six individual degrees of freedom, only two or three need to be offered to the user for control via interactive sliders. The others are implicitly defined by symmetry constraints, and should be shielded from modification by the users. This curve type can readily be generalized by allowing more than two full waves to circle the globe at the equator (Fig.3c).

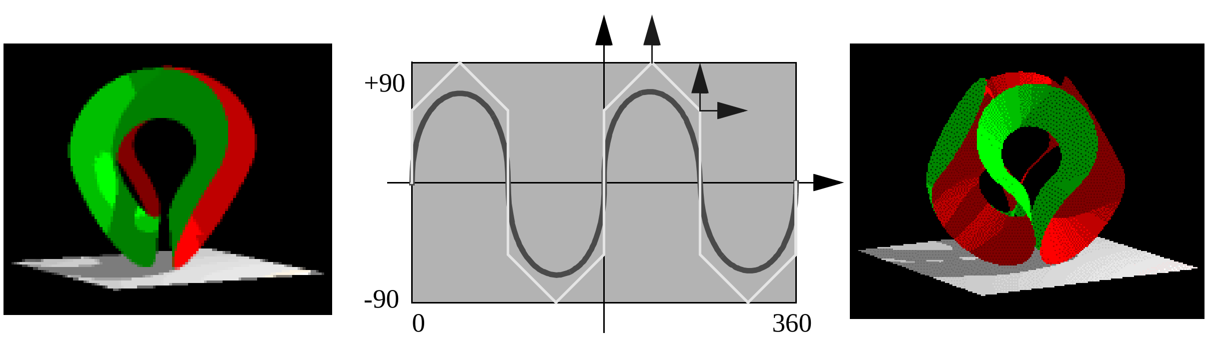

Figure 3: *Generalization of the Gabo curve: (a) 2-period band, (b) its Mercator map, (c) 3-period band.*

In a first implementation, the control points were placed at the inflection points at the equatorial crossings, one at top of the lobe controlling the approach to the pole, and two shoulder points controlling the lobe width and the pointy-ness of the lobe (Fig.3b). By watching the resulting space curve, the adjustment of these three parameters readily allows to generate a truly pleasing shape. Strictly speaking, the generated curve does not lie on the sphere, even though all control points do. For the task of creating attractive sculptures, this is not a problem. Certain deviations from the sphere may actually be desirable. For instance, I have added a parameter that controls the overall shape of the globe, either making it more pointed like a football, or flattened like the Earth's globe.

3.2 Emulating "Pax Mundi"

When I tried to see *Pax Mundi*'s pathway as a variation of the above Gabo curves, I could express it as a 4-wave circuit, with four lobes each stretching towards the North and South poles, respectively. At each pole two opposite lobes were almost touching, and the other two were significantly shorter so that they would not interfere with the primary pair. To model this in my program, I now needed three additional parameters to control the pole distance, width, and pointy-ness of the small lobes separately from the larger ones (Fig.4b).

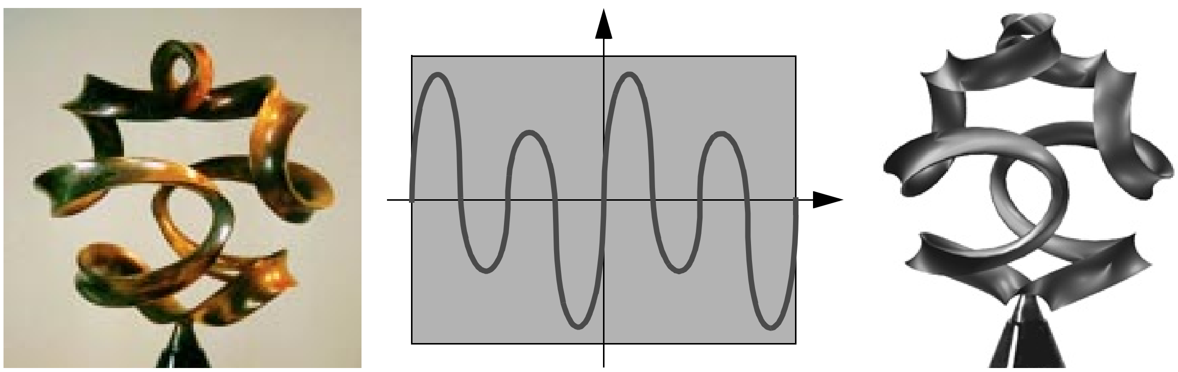

Figure 4: *Emulating Collins' Pax Mundi (a), with an amplitude modulated 4-period Gabo curve (b,c).*

A sculpture resembling *Pax Mundi* very closely can now be created by sweeping a quarter-moon-shaped cross section along this 4-period, amplitude-modulated Gabo curve. When sweeping the cross section, its azimuth, i.e. its angular rotation around the tangent vector of the curve, also has to be specified at each point. In my program, it is possible to do this for every control point of the curve -- but this would be tedious and might overwhelm the designer at the controls. I needed an automated way that would come close to an ideal solution.

Since Brent Collins strives explicitly to approximate minimal surfaces (in which the mean curvature at all points is zero), the concave part of the cross section needs to face "outwards" as it is swept around a bend. This condition has a simple and precise mathematical formulation: the symmetry axis of the cross section has to be aligned with the negative normal direction of the Frenet frame of the generator curve. Such an orientation of the cross section can easily be generated automatically for every point of the curve, and it gives good results without the need for any additional specifications. If in some bends the orientation of the cross section seems less than ideal, a small azimuth correction can still be applied locally. Of course, if the cross section is kept constant, the result is not truly a minimal surface. To balance the curvatures in the principal directions at every surface point, the concavity of the cross section would have to be adjusted to the local curvature of the sweep curve. However, Collins aims at keeping the cross section constant and actually gives it a concavity that exceeds the curvature of the sweep curve everywhere. This gives more dramatic and aesthetically more pleasing results. So I followed Collins' approach.

As I tried to approximate *Pax Mundi* as closely as possible, and in generating several other *Viae Globi* shapes with my program, I found that I needed to be able to fine-tune the positions of the tips of the lobes. Where they face one another across the poles, I wanted to adjust their relative distances from the center of the sphere and the azimuth angle by which they turn their cross sections toward one another. While this requires in principle two separate parameters, it was sufficient to just move the control point at the tip of each lobe radially in and out; this was good enough to balance the positions of adjacent lobes and the way their rims related to one another. Figure 5 gives a view of the control panel of the full *Viae Globi* generator.

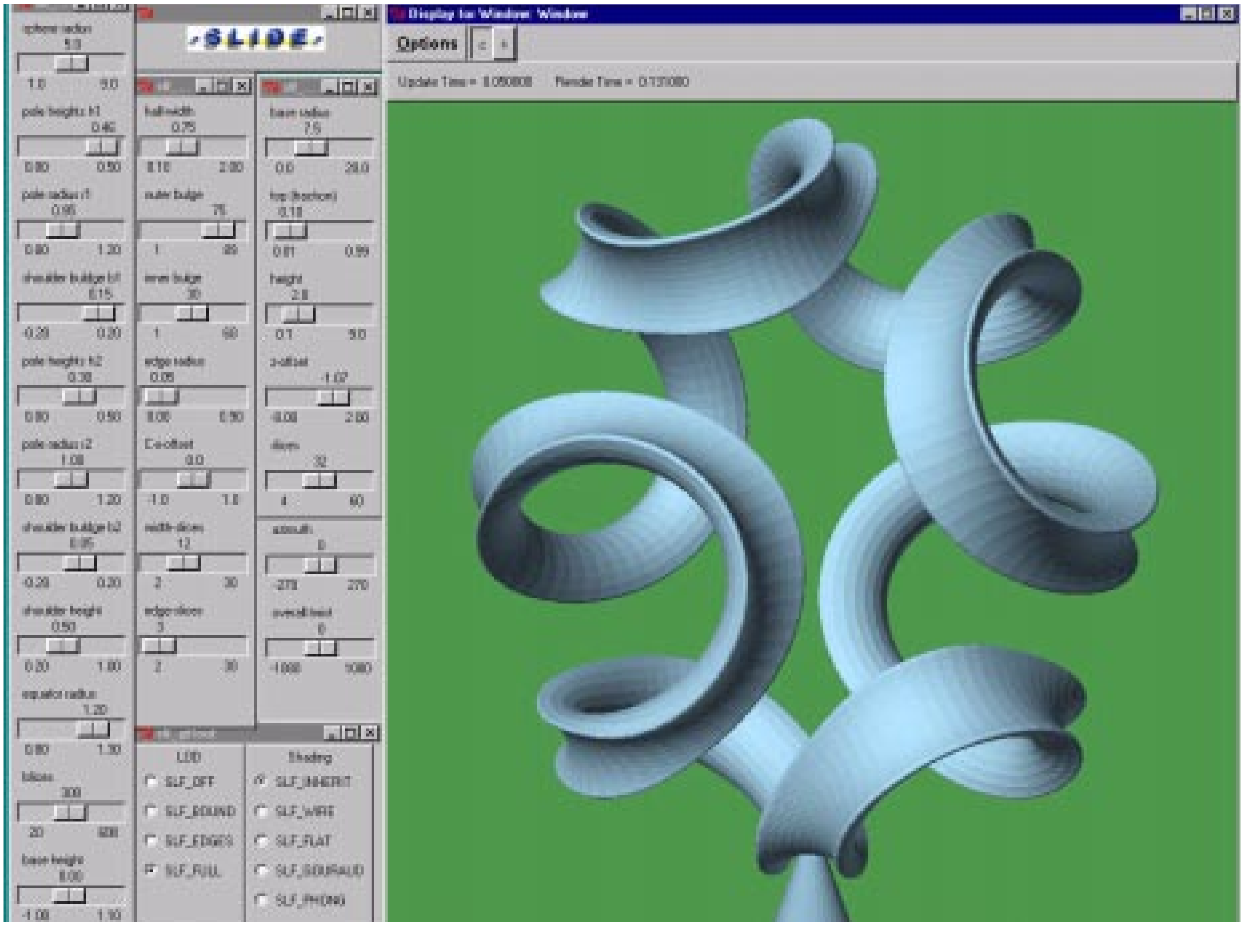

Figure 5: *The control panel of the Viae Globi sculpture generator program.*

3.3 Generalizing the Paradigm

The original curves by Gabo and Engman comprise two full waves around the equator of the globe, and *Pax Mundi* is a 4-period Gabo curve. It seems natural to ask how pleasing a sculpture one could make with three, or five, and possibly even more, full wave periods. Thus I extended the path-generating module to produce curves with from 2 to 6 periods and created the necessary controls for the size and shapes of the lobes.

In the extreme, this would require three parametric controls for every single lobe. However, I did not make each lobe completely adjustable by itself. Sculptures in which every lobe is different lack symmetry and look too random. Some level of coherence is desirable. In the sculptures with more than three periods, I grouped two or three lobes together and tied them to the same set of parameters; moreover, the amplitude pattern of the lobes at the South pole reflect the pattern at the North pole. If necessary, I alter a few lines in my program to change this grouping and thus the underlying symmetry of the sculpture. Figure 6 shows examples of maquettes with 3, 5, and 6 wave periods.

Figure 6: *(a) Via Globi 3, (b) Via Globi 5, (c) Via Globi 6*

When I tried to make a *Via Globi* sculpture with only two periods, the result looked too simple and the coverage of the sphere surface seemed to be too sparse. In Gabo's and Engman's sculptures, the two-period curves are formed by the edges of surfaces. Thus there is enough material present to give these sculptures a solid physical reality that fills the volume of the sculpture adequately. However, the narrow pathway used in *Pax Mundi* and in my other *Viae Globi* sculptures leaves too much void space. I tried to make a much broader ribbon that would fill the sphere surface adequately with just two periods, but the result seemed no longer to be part of the "roads on the globe" family; the swept cross section now became a "surface" in its own right -- more than just a band circling the surface of a sphere.

Thus I experimented with more drastic deviations from the Gabo-curve paradigm. *Via Globi 2* has two lobes at either pole, but there is an additional road passing between them. Two additional arches sweep over the two poles and connect lobes on opposite sides of the equator, leading again to a curve with six equatorial crossings as illustrated in Figure 7.

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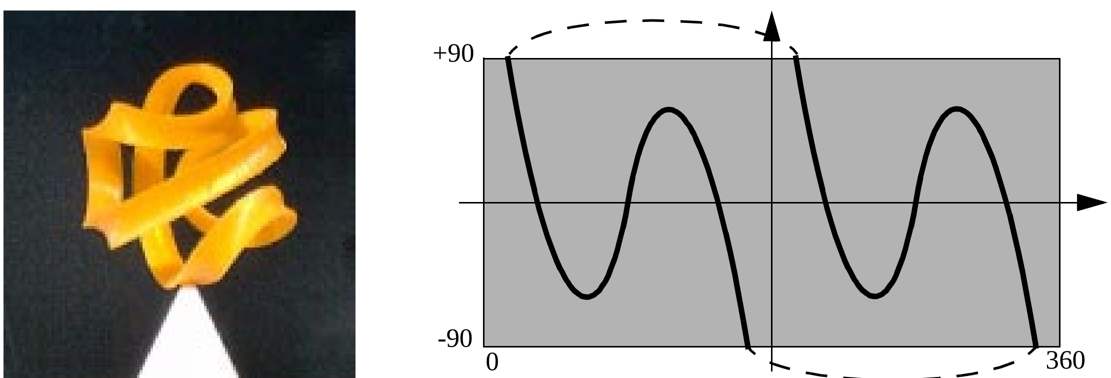

Figure 7: *(a) Via Globi 2, (b) its Mercator diagram.*

4. More General "Viae Globi"

Once the paradigm of a simple periodic Meander line along the equator of a globe gets violated, there are many more possibilities for what one can do with a loopy path on the surface of a sphere. I started by drawing curves on tennis balls, but quickly realized that I needed a more structured approach to define a path that would fill the surface of the sphere more or less evenly, ideally exhibiting some symmetry, and in the end close on itself.

4.1 Hamiltonian Paths on Platonic Solids

Since I did not have any CAD tools at my disposal that would allow me to draw nice smooth curves on a sphere I took guidance from the Platonic solids and used their vertices to place control points for B-splines in some spherically symmetrical ways. The Via Globi structures discussed in the previous section all have a dominant symmetry axis through the two poles of the globe and perhaps some additional two-fold rotational axes through points on the equator. What I was now looking for were paths with more truly spherical symmetries as exhibited by the Platonic solids, which have higher-order symmetry axes in more than two opposite directions in space. A first possibility that occurred to me, was to place the control points for the B-spline paths at the vertices of one of these highly regular polyhedra, and connect them into a closed path of some symmetry. This path need not be strictly Hamiltonian; the same vertices can be visited more than once as part of loops embedded in adjacent faces, since the approximating nature of the B-spline will pull the two paths sufficiently far away from one another, so that the generated roads would not interfere.

The first attempts with this new approach to defining the sweep path did not cover much new ground. By starting with a cube, I readily reproduced a curve that corresponded to Via Globi 2 when I connected all eight vertices in a true Hamiltonian path. In another path that encircled all six faces in sequence, the emerging curve was the 3-period Gabo curve. Attempts with an octahedron also reproduced the 3-period Gabo curve. Working with the icosahedron and the dodecahedron produced starts of promising looking patterns, but then did not lead to closed curves. In the few instances where it did, the result was the 5-period Gabo curve. When I tried to place small pieces of S-shaped curves onto these "spherical" polyhedra, and then started to connect them while maintaining some symmetry, the resulting pattern always consisted of more than one closed loop.

I finally realized, that placing a single closed loop on a sphere is an operation that cannot create more symmetry than that exhibited by a single great circle. Other than the dominant axis of infinite rotational symmetry, all other curve points have at most C_2 rotational symmetry. By adding undulations to the starting loop, the dominant symmetry can be reduced to C_n or D_n with a value *n* that corresponds to the number of wave periods introduced. If I was looking for points with C_3 or higher symmetries, these points had to occur in locations where three or more lobes were congregating. Placing partial lobes into appropriate places leads to the difficult task of then connecting all the loose ends. Occasionally I succeeded, but always found one of the simpler patterns described in the last section. For instance, when I placed C_3 points with three lobes at the vertices of a tetrahedron, the only single-loop, non-self-intersecting, closed curve I found turned out to be a 6-period Gabo curve. It became clear, that it is better to start from a simple closed curve and gradually deform it to find new and interesting pathways. But for that, I needed better tools!

4.2 Sticking to the Sphere

What I needed was an easy way of making more free-form curves that lie precisely embedded in the surface of a sphere. The topic of spherical splines has been addressed in several papers at recent graphics conferences, since it is also important for generating smooth camera paths around an object. Whenever such paths travel over the poles of the sphere, the singularities in any coordinate system based on Euler angles become problematic. A coordinate system without such singularities is offered by the set of unit quaternions, and several papers have described the generation of smooth paths on a sphere using such a formulation. These approaches still are not totally satisfactory. They do not handle very well the cases of interpolating curves through widely separated control points, where a pair of subsequent points may lie on almost opposite sides of the sphere. Moreover, quaternion schemes often inherit an undesirable property: since quaternions are non-commutative, front-to-back symmetry is lost. If we use the same set of control points, but construct the associated spline curve in the reverse direction, the geometry of the curve may be different from the one constructed in the forward direction.

Around that time, Jane Yen and I were also developing Escher-like tilings on the sphere [12]. In this context we needed the capability to draw smooth curves on spheres. Thus we set out to develop a simple editor tool for smooth splines on spherical surfaces that would not have any of the above shortcomings. We ended up with two very usable paradigms. The first one simply takes the classical de Casteljau algorithm and implements it with great circular arcs on the sphere rather than with straight linear interpolation in Euclidean space. It yields very robust and pleasing approximating curves on the sphere. Since all new subdivision points generated in every step always lie on the sphere, the whole final curve is truly embedded in this surface. With this tool, the designer can place an arbitrary sequence of control points on the sphere and then produce a very smooth approximating path on the sphere (Fig.8a). The interactive speed of the program allows the user to move the control points until the generated spline curve is aesthetically satisfactory.

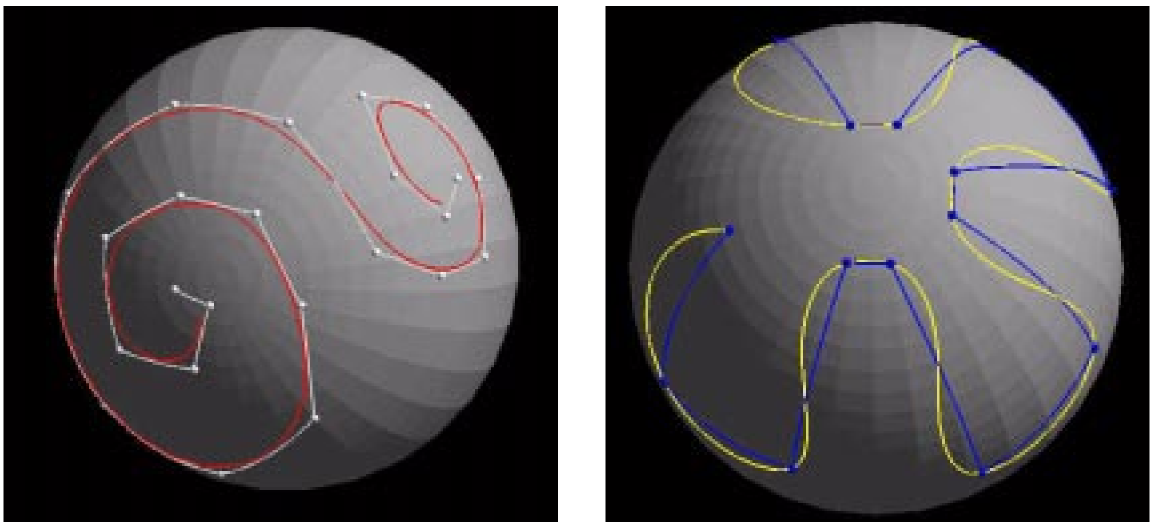

Figure 8: *Editor tools for generating (a) approximating or (b) interpolating curves.*

The second approach generates interpolating curves through the given set of points (Fig.8b). It uses a subdivision scheme that starts from the piece-wise linear control polygon and in each generation finds new midpoints for all line segments on their respective perpendicular bisectors. The exact position of any such new curve point is found by blending two circles. One circle is constructed through the end-points of the segment to be subdivided and also through the preceding control point of the curve; the other one passes through the subsequent control point, instead. The new subdivision point is then chosen so that it averages the turning angles implied by the two circles. This approach results in very pleasing loopy curves -- exactly the kind that I had been trying to generate with B-splines by carefully adjusting all their control points. This interpolation scheme will be described in more details in a future paper.

A key motivation for producing sweep curves that lie truly on the surface of a sphere was the hope that it would no longer be necessary to adjust the radial position of the control points of the ends of each lobe to bring the edges of adjacent sweeps into alignment. By following bends that would lie exactly in the spherical surface I hoped to obtain automatic alignment, as well as a graceful and smooth behavior of the azimuth angle as the sweep would turn around a narrow bend. This is another step in the continuing quest to create the smallest most productive set of parameters that can generate the widest variety of aesthetically pleasing shapes with the least amount of fine-tuning effort. Figure 9 shows that the latest step in automating the fine-tuning of the lobe tips has been successful. The sudden jerk in azimuth angle sometimes observed on the earlier Gabo curves is now gone, and adjacent flanges belonging to two different bends, nicely align themselves to each other.

Figure 9: *Sweep along a truly spherical path.*

4.3 Path Deformations

Since constructing an interesting and novel path by connecting the vertices of a Platonic solid into a closed chain is difficult, I tried a different approach. I started with a closed path on the sphere and gradually deformed it into a more sophisticated curve -- possibly maintaining certain symmetry constraints. With the new editing tools for curves on a sphere, this is an easy and enjoyable task.

In a first experiment, I started from a 2-period Gabo curve -- the one that looks like the seam on a tennis ball. Where opposite lobes approach one another, I created indentations that I gradually enlarged until there were four full lobes coming together at the poles. Then I adjusted the rest of the curve to give a more or less uniform coverage of the sphere surface and to achieve a pleasing, gradual change in geodesic curvature along the whole pathway (Fig.10a).

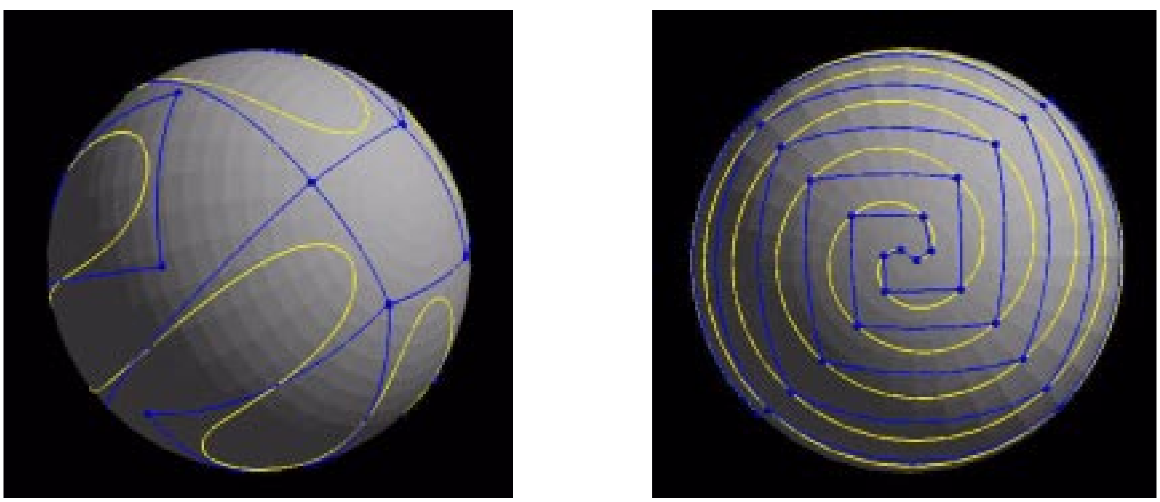

Figure 10: *Results of deforming a simple closed loop on the sphere.*

Another successful experiment started from a great circle passing through both poles. Then I applied opposite twists at both poles and gradually deformed the original loop into a double spiral. With a twist of about 180° between both poles, the curve shows similarity to the 2-period Gabo curve; with an overall twist of 360°, it has the basic shape of *Via Globi 2*. Incidentally, this is also the basic pattern that Collins used in his *Egg* sculpture, except that in this latter case the sweep curve is mapped onto an elongated ellipsoid [3]. Continuing the twisting of the meridian results in the curve shown in Figure 10b.

5. Conclusions

The focus of my collaboration with Brent Collins has been to understand and express some of his sculptures as procedural recipes captured in the form of a computer program. For some of his work this effort is particularly appropriate, since Collins often thinks about his sculptures in terms of a geometrical generating principle, such as curves hugging the surface of spheres or ellipsoids, or, alternatively, spiraling around one another along a well-defined path in space. By capturing such a concept in the right way and by judiciously introducing extra geometrical parameters, some of Collins' visionary shapes cannot only be reproduced fairly closely, but they could then be fine-tuned in search for even more optimal realizations. An additional payoff from this approach is, that through some small changes to the underlying recipe, new forms can be created that Collins has not (yet) had the time to build. Through the miracles of free-form fabrication, small maquettes of such new sculptures can be generated in a matter of days.

But the most exciting prospect is, that through playful interaction with the generator program new ideas will emerge, that can be realized very quickly by just changing a few lines in the code. In this way the original concept behind a particular structure can be extended, and new forms can be produced that neither Collins nor I were thinking of at the time of the construction of the initial inspirational piece or its capture in the form of a computer program. With a tailor-made visualization program, the computer becomes an amplifier and accelerator for a designer's creative powers. New ideas can be quickly brought to a state, where they can be visualized with enough geometrical detail so that their aesthetic merits can be judged. Promising shapes can be prototyped as plastic maquettes, and the truly successful ones can be scaled up and realized through investment casting in a more permanent form. This paper has documented this process for the *Viae Globi* series of sculptures inspired by the Brent Collins' exquisite sculpture entitled *Pax Mundi*. A small bronze cast of *Via Globi 3* is shown in Figure 11..

Figure 11: *Via Globi 3, bronze, four inches tall, 2000.*

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