1. Suppose that n labeled balls are thrown independently and uniformly at random into n labeled bins.

   (a) Find the conditional probability that bin 1 has one ball given that exactly one ball fell into the first three bins.

   (b) Find the conditional expectation of the number of balls in bin 1 under the condition that bin 2 received no balls.

2. You are dealt a hand of five cards from a randomly shuffled deck.

   (a) What is the probability of getting a straight?

   (b) Conditioned on getting a straight, what is the probability that the hand contains an ace?

3. Consider a random permutation on the set $\{1, 2, \ldots, n\}$. What is the expected number of cycles of length k?

4. Here are some additional problems based on the randomized min-cut algorithm discussed in class (MU Section 1.4). We have seen in class that the probability that the algorithm terminates with a specific min-cut is at least $\frac{2}{n(n-1)}$ and you are asked to show on the homework that any graph has at most $\frac{n(n-1)}{2}$ distinct min-cuts. In this problem, we see that both bounds are tight.

   (a) Give a family of connected undirected graphs G_n on n vertices such that G_n has exactly $\frac{n(n-1)}{2}$ distinct min-cuts.

   (b) Give a family of connected undirected graphs G_n on n vertices and a min-cut S_n of G_n such that the probability that the randomized min-cut algorithm on input G_n terminates with the cut S_n is exactly $\frac{2}{n(n-1)}$.

1. (a) $1/3$.

   (b) $\frac{n}{n-1}$.

2. (a) $\frac{10 \cdot 4^5}{\binom{52}{5}}$. (Both `A 1 2 3 4 5` and `10 J Q K A` are straights.)

   (b) $\frac{1}{5}$.

3. For each i, let Y_i denote the indicator variable that equals 1 if i lies in a cycle of length k, and 0 otherwise. Then, $\Pr[Y_i = 1] = \frac{1}{n}$. This is because the number of cycles of length k that contains i is $(n-1)(n-2)\cdots(n-k+1)\cdot(n-k)! = (n-1)!$ (i can be mapped to any of $n-1$ values, and that can be mapped to any of $n-2$ values, ...; the $n-k$ elements that are not part of the cycle can correspond to any permutation). It follows that $E[Y_i] = \frac{1}{n}$.

   Next, let Z_k be the random variable for the number of cycles of length k. We claim that:

   $$Y_1 + Y_2 + \ldots + Y_n = k \cdot Z_k$$

   This means that the above expression holds for every point in the sample space, namely, for every permutation on $\{1, 2, \ldots, n\}$. Indeed, fix any such permutation π. Every cycle of length k in π contributes a value k to the expression on the left, because each element in the cycle contributes a value 1. By linearity of expectations, we have:

   $$E[Y_1] + E[Y_2] + \ldots + E[Y_n] = E[k \cdot Z_k] = k \cdot E[Z_k]$$

   Hence, $k \cdot E[Z_k] = n \cdot 1/n$ and thus $E[Z_k] = \frac{1}{k}$.

   Now, if we want to compute the expected number of cycles in a random permutation, that is simply $E[Z_1] + \cdots + E[Z_n] = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \Theta(\log n)$.

4. For both parts, we can simply take G_n to be a cycle of length n. The min-cut of G_n has size 2, and every subset of 2 distinct edges of G_n is a min-cut. Therefore, there are exactly $\frac{n(n-1)}{2}$ distinct min-cuts.

   One way to solve (b) is to consider a cut C_n corresponding to 2 adjacent edges. In order that the min-cut terminates with this cut, the algorithm must not contract either of the 2 edges during any iteration. The probability that this happens in the first iteration is $\frac{n-2}{n}$, $\frac{n-3}{n-1}$ in the second iteration, and $\frac{1}{3}$ in the final iteration. Multiplying these probabilities yields $\frac{2}{n(n-1)}$.

   A slick argument for (b) is as follows: the min-cut algorithm terminates with each min-cut with probability at least $\frac{2}{n(n-1)}$. There are $\frac{n(n-1)}{2}$ min-cuts, so it must be that the algorithm terminates with each of these min-cuts with probability exactly $\frac{2}{n(n-1)}$; otherwise, the sum is more than 1, a contradiction.