1. Suppose we only have access to a standard unbiased coin. Describe an algorithm to pick a number from $\{1, 2, \ldots, 174\}$ u.a.r. such that the expected number of coin tosses is not more than 12.

2. You need a new staff assistant and you have n people to interview. You want to hire the best candidate for the position. You interview the candidate one by one. After you interview the kth candidate, you either offer the candidate the job before the next interview or you forever lose the chance to hire that candidate. We suppose the candidates are interviewed in a random order (from all $n!$ possible orderings).

   We consider the following strategy. First, interview m candidates but reject them all. After that, hire the first candidate who is better than the first m candidates. Show that the probability that we hire the best candidate is:

$$\frac{m}{n} \sum_{j=m+1}^{n} \frac{1}{j-1}$$

   Note that this probability is approximately $\frac{m}{n}(\ln n - \ln m) = 1/e$ when $m = n/e$.

   (Hint: partition the sample space into mutually exclusive events, and think about the analysis of the quicksort algorithm presented in class.[1])

3. Suppose we flip a fair coin n times to obtain n random bits. Consider all $m = \binom{n}{2}$ pairs of these bits in some order. Let Y_i be the exclusive-or of the i pair of bits, and let $Y = \sum_{i=1}^{m} Y_i$ be the number of Y_i that equal 1.

   (a) Show that each Y_i is 0 with probability $1/2$ and 1 with probability $1/2$.

   (b) Show that the Y_i's are not mutually independent.

   (c) Compute $\text{Var}(Y)$.[2]

[1] Let E_j denote the event that the best candidate is the jth candidate. Conditioned on E_j, we hire the best candidate iff the top candidate amongst the first $j-1$ candidates is amongst the first m candidates. This occurs with probability $\frac{m}{j-1}$ for $j > m$.

[2] It is the case that for all $i \neq j$, $E[Y_i Y_j] = \Pr[Y_i Y_j = 1] = \frac{1}{4}$ (there are two cases to check, depending on whether the pairs corresponding to Y_i, Y_j are disjoint). It is then easy to check that $\text{Var}(Y) = \frac{m}{4}$.