1. Show that if a graph G has n vertices and m edges, then there exists a cut of size at least $\frac{m(n+1)}{2n}$.

   [HINT: The uniform distribution over all partitions implies the existence of a cut of size at least $m/2$. Try a different distribution over partitions.]

2. In this problem, a 2-coloring of a graph G is an assignment of colors red or green to the *edges* of G. Consider a random 2-coloring of K_n (the complete graph on n vertices), namely each edge is assigned a random color.

   (a) Compute the expected number of monochromatic subgraphs K_k in K_n (for a fixed k and n).

   (b) Compute n as a function of k such that the expected number of monochromatic K_k is 1 (the asymptotic value suffices). How about k as a function of n?

   We have seen in lecture that if $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$, then there exists a 2-coloring of K_n so that it has no monochromatic K_k. Here, we generalize the result to handle the case $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} \geq 1$ (where we color a smaller graph $K_x, x \leq n$).

   (c) Show that there exists a 2-coloring of K_n such that the number of monochromatic K_k is at most $\left\lfloor \binom{n}{k} \cdot 2^{1-\binom{k}{2}} \right\rfloor$.

   (d) Deduce that there exists a 2-coloring of K_x so that it contains no monochromatic K_k, where

   $$x = n - \left\lfloor \binom{n}{k} \cdot 2^{1-\binom{k}{2}} \right\rfloor .$$

   [NOTE: Note that $x = n$ whenever $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$.]