1. Consider a Markov chain on the line with states $\{0, 1, \ldots, n\}$, with $P_{0,1} = P_{1,2} = \cdots = P_{n-1,n} = P_{1,0} = P_{2,1} = \cdots = P_{n,n-1} = P_{0,0} = P_{n,n} = \frac{1}{2}$.

   (a) Show that the process is irreducible and aperiodic. What is the stationary distribution? [1]

   (b) Consider a random walk that starts at state 1. What is the probability that it reaches 0 *before* reaching n? [2]

   (HINT: Relate this problem to the gambler's ruin problem.)

2. A cat and a mouse each independently take a random walk on a connected undirected non-bipartite graph G with n vertices and m edges. They start at the same time on different nodes, and each makes one transition at each time step. The cat eats the mouse if they are ever at the same node at some time step. Show that the expected time before the cat eats the mouse is $O(m^2 n)$.[3]

   (HINT: Consider a Markov chain whose states are the ordered pairs (a, b) where a is the position of the cat and b is the position of the mouse. You would need to recall some basic facts on hitting times.)

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[1] You should be able to verify that the uniform distribution over the $n+1$ states is indeed a stationary distribution.

[2] The answer is $1 - \frac{1}{n}$ (as expected, the answer is close to 1). This is the same as the probability that the first player wins in a fair gambling situation between two players with $n-1$ dollars and 1 dollar respectively (MU Section 7.2.1).

[3] The Markov chain in the hint is a random walk on a connected undirected non-bipartite graph G' with n^2 vertices and m^2 edges. Suppose the initial positions of the cat and the mouse are a_0 and b_0 respectively. Since G is non-bipartite, there exists a path in G of even length connecting a_0 and b_0. Moreover, the path has length at most n. This yields a path of at most n in G' connecting (a_0, b_0) and (a_0, a_0). By an analysis similar to the proof of MU Lemma 7.16, this yields an upper bound of $2m^2 n$ (with a contribution of $2m^2$ from each edge on the path) on the expected number of steps to reach (a_0, a_0) from (a_0, b_0) (i.e., the hitting time between those two vertices).