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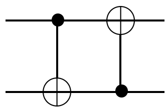
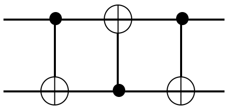
Problem Set #2

Quantum Gates and Circuits

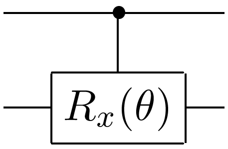
1. **Fun with controlled-NOT gates**: What does the following circuit do?



How about this one?



2. **Controlled-rotation gates**: This gate,

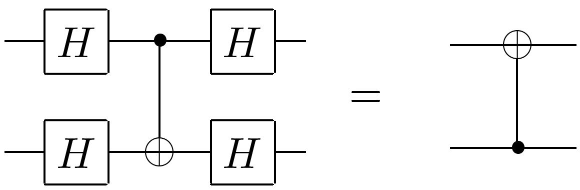


rotates the target (lower) qubit by $R_x(\theta)$ (angle θ about the $\hat{x}$ axis, if the control (upper) qubit is $|1\rangle$, otherwise it does nothing. The unitary transform can be written in matrix form as

$$\Lambda_1(R_x(\theta)) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ 0 & 0 & i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \tag{1}$$

Show how this gate can be implemented using two controlled-NOT gates and two single qubit rotations.

3. **Basis transformations**: Unlike ideal classical gates, ideal quantum gates do not have high-impedance inputs. In fact, the role of "control" and "target" are arbitrary – they depend on what basis you think of a device as operating in. We have given the truth table for a CNOT and shown how the control qubit does not get changed in the classical 00, 01, 10, 11 basis. However, in reality, the control qubit *does* change: its phase is flipped depending on the state of the "target" qubit! Show that



4. **Pauli Matrices**: The Pauli matrices σ_x, σ_y, and σ_z are very important in the mathematics of quantum computation, because they are generators for the group of single-qubit transformations. Mathematically, this is the $SU(2)$ group, of 2×2 complex unitary matrices (those for which $(M^T)^* \equiv M^\dagger = M^{-1}$) with determinant $+1$, which describe solid body rotations in three dimensions.

   a. Using

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{3}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

   and recalling that

$$e^{i\lambda M} = \sum_k \frac{(i\lambda)^k}{k!} M^k \tag{5}$$

   show that

$$R_x(2\theta) \equiv e^{i\theta\sigma_x} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \tag{6}$$

$$R_y(2\theta) \equiv e^{i\theta\sigma_y} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{7}$$

$$R_z(2\theta) \equiv e^{i\theta\sigma_z} = \begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix} \tag{8}$$

   It is useful to note that $\sigma_i^2 = I$, the identity matrix.

   b. Suppose you want $R_z(\theta)$, but only have gates which perform $R_x(\pi)$ and $R_y(\theta)$. Show how to compose these gates to accomplish the desired rotation.

   c. The Pauli matrices have wonderful commutation properties which are important in their role as the generators of $SU(2)$. Show that

$$\sigma_x\sigma_y - \sigma_y\sigma_x = 2i\sigma_z \tag{9}$$

$$\sigma_y\sigma_z - \sigma_z\sigma_y = 2i\sigma_x \tag{10}$$

$$\sigma_z\sigma_x - \sigma_x\sigma_z = 2i\sigma_y\,. \tag{11}$$

   Also show that

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 0\,, \tag{12}$$

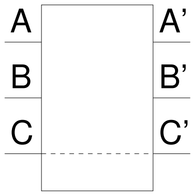
for $i \neq j$. These *commutation* and *anti-commutation* relations are fundamental to the Lie Algebra for $SU(2)$. They are often written concisely as

$$\begin{aligned} [\sigma_i, \sigma_j] &= 2i\epsilon_{ijk}\sigma_k \qquad (13) \\ \{\sigma_i, \sigma_j\} &= 2\delta_{ij} I\,, \qquad (14) \end{aligned}$$

where ϵ_{ijk} is a totally antisymmetric pseudovector known as the Levi-Civita symbol.

5. **Fredkin and the Controlled-NOT gate**: The Fredkin gate has the truth table

Inputs			Outputs		
A	B	C	A'	B'	C'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	1	0	1
1	0	0	1	0	0
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	1	1	1



The classical controlled-NOT gate is also reversible, albeit with only two inputs and two outputs.

a. Construct a controlled-NOT gate from two Fredkin gates.

b. Show that the controlled-NOT gate (which can also be configured to work as a simple NOT gate) is not a universal gate.

6. (extra credit) **Quantum Fredkin Gate**: Construct a quantum version of the Fredkin gate using only six two-qubit gates.

7. **Hadamard transform of two strings**: Suppose that x and y are two m-bit strings. Calculate the Hadamard transform

$$H_m \left[\frac{|x\rangle + |y\rangle}{\sqrt{2}} \right] \qquad (15)$$