## Linear Programming Continued

### 1. A production scheduling example

We have the demand estimates for our product for all months of 1999, $d_i : i = 1, \ldots, 12$, and they are very uneven, ranging from 440 to 920. We currently have 30 employees, each of whom produce 20 units of the product each month at a salary of 2,000; we have no stock of the product. How can we handle such fluctuations in demand? Three ways:

- overtime —but this is expensive since it costs 80% more than regular production, and has limitations, as workers can only work 30% overtime.
- hire and fire workers —but hiring costs 320, and firing costs 400.
- store the surplus production —but this costs 8 per item per month.

This rather involved problem can be formulated and solved as a linear program. As in all such reductions, a crucial first step is defining the variables:

- Let $w_i$ be the number of workers we have the $i$th month —we have $w_0 = 30$.
- Let $x_i$ be the production for month $i$.
- $o_i$ is the number of items produced by overtime in month $i$.
- $h_i$ and $f_i$ is the number of workers hired/fired in the beginning of month $i$.
- $s_i$ is the amount of product stored at the end of month $i$.

We now must write the constraints:

- $x_i = 20w_i + o_i$ —the amount produced = regular production + overtime production.
- $w_i = w_{i-1} + h_i - f_i, w_i \geq 0$ — new workers = old workers + hired - fired.
- $s_i = s_{i-1} + x_i - d_i \geq 0$ —the amount stored at the end of this month is what we started with, plus the production, minus the demand.
- $0 \leq o_i \leq 6w_i$ —only 30% of items produced in overtime.

Finally, what is the objective function? It is

$$\min 2000 \sum w_i + 400 \sum f_i + 320 \sum h_i + 8 \sum s_i + 180 \sum o_i.$$

### 2. Reductions between versions of linear programming:

The general linear programming may involve constraints that are equations, or inequalities in either direction. Its variables may be nonnegative, or could be unrestricted in sign. And we may be either minimizing or maximizing a linear function. It turns out that we can easily translate any such version to any other. One such translation that is particularly useful is from the general form to the one required by simplex: *minimization, nonnegative variables, and equations.*

To turn a maximization problem into a minimization one, we just multiply the objective function by $-1$.

To turn an inequality $\sum a_i x_i \leq b$ into an equation, we introduce a new variable $s$ (the *slack variable* for this inequality), and rewrite this inequality as $\sum a_i x_i + s = b, s \geq 0$. Similarly, any inequality $\sum a_i x_i \geq b$ is rewritten as $\sum a_i x_i - s = b, s \geq 0$; $s$ is now called a *surplus* variable.

We handle an unrestricted variable $x$ as follows: We introduce two nonnegative variables, $x^+$ and $x^-$, and replace $x$ by $x^+ - x^-$. This way, $x$ can take on any value.